IN Variant distributions supported on the nilpotent cone of a semisimple Lie algebra

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Abstract. Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \) and \( \mathcal{D}(g) \) be the algebra of differential operators with polynomial coefficients on \( g \). If \( g_0 \) is a real form of \( g \), we give the decomposition of the semisimple \( \mathcal{D}(g)^G \)-module of invariant distributions on \( g_0 \) supported on the nilpotent cone.

0. Introduction

Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \). Choose a Cartan subalgebra \( h \) of \( g \) and let \( W \) be the associated Weyl group. Denote by \( W^\vee \) the set of isomorphism classes of irreducible \( W \)-modules and by \( \mathcal{H}(h^\vee) \) the graded vector space of \( W \)-harmonic polynomials on \( h \). For \( \chi \in W^\vee \), set \( b(\chi) = \inf \{ j \in \mathbb{N} : [H^j(h^\vee) : \chi] \neq 0 \} \) and choose a \( W \)-submodule \( V_\chi \subset H^{b(\chi)}(h^\vee) \) in the class of \( \chi \). Denote by \( d(\chi) \) the dimension of \( V_\chi \).

Let \( S(g^\vee) \) be the algebra of polynomial functions on \( g \) and \( \mathcal{D}(g) \) be the algebra of differential operators on \( g \), with coefficients in \( S(g^\vee) \). The group \( G \) acts on \( g \) via the adjoint action, and hence has an induced action on \( S(g^\vee) \), \( S(g) \) and \( \mathcal{D}(g) \). Denote the differential of this action by \( \tau : g \to \mathcal{D}(g) \). Let \( S_+(g)^G \) and \( S_+(g^\vee)^G \) be the set of invariant elements without constant term. Recall that \( N(g) \), the nilpotent cone of \( g \), is the variety of zeroes of the ideal \( S_+(g^\vee)^G S(g^\vee) \).

Let \( g_0 \) be a real form of \( g \) with adjoint group \( G_0 \subset G \). Denote by \( \text{Db}(g_0) \) the \( \mathcal{D}(g) \)-module of distributions on \( g_0 \). Then, the subspace of invariant distributions \( \text{Db}(g_0)^G_0 = \{ T \in \text{Db}(g_0) : \tau(g).T = 0 \} \) is a \( \mathcal{D}(g)^G \)-module, containing the submodule of invariant distributions supported on the nilpotent cone

\[
\text{Db}(g_0)^G_0 |_{nil} = \{ \Theta \in \text{Db}(g_0)^G_0 : \text{Supp} \Theta \subset N(g_0) \}
\]

where \( N(g_0) = N(g) \cap g_0 \) is the nilpotent cone of \( g_0 \). The structure of \( \text{Db}(g_0)^G_0 |_{nil} \) as a vector space is well understood, see, for example, \([1, 5]\). Let \( [h_1], \ldots, [h_r] \) be the conjugacy classes of Cartan subalgebras of \( g_0 \). For each \( j \), let \( \varepsilon_{l,j} : W(h_j) \to \{ \pm 1 \} \) be the imaginary signature of the real Weyl group \( W(h_j) \). Then \([5, \text{Proposition 6.1.1}]\) there exists a vector space isomorphism

\[
\bigoplus_{j=1}^r S(h_j, \mathbb{C})^{\varepsilon_{l,j}} \cong \text{Db}(g_0)^G_0 |_{nil}
\]
where $S(\mathfrak{h},C)^{\varepsilon_{I,j}}$ is the isotypic component of type $\varepsilon_{I,j}$ in the $W(\mathfrak{h})$-module $S(\mathfrak{h},C)$.

One aim of this note is to give a complete description of the $\mathcal{D}(\mathfrak{g})^G$-module $\mathbb{D}(\mathfrak{g})^G_{\text{nil}}$. This description is given in terms of the simple summands of the equivariant holonomic $\mathcal{D}(\mathfrak{g})$-module

$$\mathcal{M} = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S(\mathfrak{g}^*)^G).$$

By [9], [18] or [13], it is known that we have a decomposition

$$\mathcal{M} = \bigoplus_{\chi} \tau(\chi) \mathcal{M}_\chi$$

where $\mathcal{M}_\chi$ are pairwise non-isomorphic simple $\mathcal{D}(\mathfrak{g})$-modules. Moreover, the support in $\mathcal{M}_\chi$ of $\chi$ is the closure of a nilpotent orbit and $\mathcal{M}_\chi^G$ is a simple $\mathcal{D}(\mathfrak{g})^G$-module. Then we have, see Corollary 3.6:

**Theorem A.** The $\mathcal{D}(\mathfrak{g})^G$-module $\mathbb{D}(\mathfrak{g})^G_{\text{nil}}$ decomposes as

$$\mathbb{D}(\mathfrak{g})^G_{\text{nil}} \cong \bigoplus_{\chi \in W^{-}} m_\chi \mathcal{M}_\chi^G$$

This theorem is proved by combining the isomorphism (*) and the properties, established in [18, 11, 12, 13], of the Harish-Chandra homomorphism

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W.$$

In the particular case where $\mathfrak{g}_0$ is a complex Lie algebra $\mathfrak{g}_1$ (viewed as a real Lie algebra), Theorem A was proved by N. Wallach [18]. In this case, $\mathfrak{g} \simeq \mathfrak{g}_1 \times \mathfrak{g}_1$, $W \simeq W_1 \times W_1$ where $W_1$ is the Weyl group of $\mathfrak{g}_1$. Then, each $M_\chi$ occurring in the decomposition of $\mathbb{D}(\mathfrak{g})^G_{\text{nil}}$ is of the form $M_\phi \boxtimes M_\psi$ with $\chi = \phi \boxtimes \psi$, $\phi \in W_1^-$, and one has $m_\chi = 1$. Hence $\mathbb{D}(\mathfrak{g})^G_{\text{nil}} \cong \bigoplus_{\phi \in W_1^\circ} M^G_{\phi} \boxtimes M^G_{\phi'}$ as a $\mathcal{D}(\mathfrak{g})^G$-module.

The next corollary is an easy consequence of Theorem A.

**Corollary B.** Let $\chi \in W^-$. Then, $M_\chi \cong \mathcal{D}(\mathfrak{g}).\Theta$ for some $\Theta \in \mathbb{D}(\mathfrak{g})$ if, and only if, $V_\chi^{\varepsilon_{I,j}} \neq 0$ for some $j \in \{1, \ldots, r\}$.

In Remark 3.7, we apply this result to give examples of modules $\mathcal{M}_\chi$ which cannot be generated by a distribution on any real form of $\mathfrak{g}$.

1. **Preliminary results**

We retain the notation of the introduction. Denote by $\Delta$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and fix a system $\Delta^+$ of positive roots. Set $n = \dim \mathfrak{g}$, $\ell = \dim \mathfrak{h}$ and $\nu = \# \Delta^+$, hence $n = 2\nu + \ell$. Let $\pi$ be the product of positive roots and recall that $x \in \mathfrak{g}$ is called generic if $\pi(x) \neq 0$. If $a \subset \mathfrak{g}$, we denote by $a^*$ the set of generic elements in $a$.

For $q \in S(\mathfrak{g})$, let $\partial(q) \in \mathcal{D}(\mathfrak{g})$ be the corresponding differential operator with constant coefficients. Let $\{e_i\}_{1 \leq i \leq n}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form $\kappa$ such that $\{e_i\}_{1 \leq i \leq \ell}$ is a basis of $\mathfrak{h}$. Denote by $x_i \in S(\mathfrak{g}^*)$, $1 \leq i \leq n$, the associated coordinate functions; thus $\partial(e_i)$ identifies with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. Denote the Euler vector fields on $\mathfrak{g}$ and $\mathfrak{h}$ by $E_\mathfrak{g} = \sum x_i \partial_i$ and $E_\mathfrak{h} = \sum x_i \partial_i$.

We now give some notation and results from [11, 12, 13, 18]. Recall first that the algebra homomorphism, defined by Harish-Chandra,

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$$
extends the Chevalley isomorphisms $S(\mathfrak{g})^G \cong S(\mathfrak{h})^W$ and $S(\mathfrak{g}^*)^G \cong S(\mathfrak{h}^*)^W$. The map $\delta$ is surjective and its kernel is $\mathcal{I} = (\mathcal{D}(\mathfrak{g})^\tau(\mathfrak{g}))^G$. This enables one to identify, through $\delta$, modules over $A(\mathfrak{g}) := \mathcal{D}(\mathfrak{g})^G/\mathcal{I}$ with $\mathcal{D}(\mathfrak{h})^W$-modules.

**Lemma 1.1.** Let $D \in \mathcal{D}(\mathfrak{g})^G$. Then $D = P + Q$ with $P \in \mathbb{C}(S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G)$ and $Q \in \mathcal{I}$.

*Proof.* By [11], we know that $\mathcal{D}(\mathfrak{h})^W = \mathbb{C}(S(\mathfrak{h})^W, S(\mathfrak{h}^*)^W)$. The lemma is therefore consequence of the properties of $\delta$ previously recalled. \hfill \Box

Recall that the $(\mathcal{D}(\mathfrak{h})^W, W)$-module $S(\mathfrak{h}^*)$ decomposes as

\[(1.1) \quad S(\mathfrak{h}^*) \cong \bigoplus_{\chi \in W^*} V_{\chi} \otimes_{\mathbb{C}} V_{\chi} \]

where $V_{\chi} = \text{Hom}_{\mathcal{D}(\mathfrak{h})^W}(V_{\chi}, S(\mathfrak{h}^*))$ is a simple $\mathcal{D}(\mathfrak{h})^W$-module. Let $\{v_{\chi}^1, \ldots, v_{\chi}^d(\chi)\}$ be a basis of $V_{\chi}$, then $V_{\chi} \cong \mathcal{D}(\mathfrak{h})^W.v_{\chi}^j$ for all $j$ and (1.1) implies that

\[S(\mathfrak{h}^*) = \bigoplus_{\chi \in W^*} \bigoplus_{j = 1}^d \mathcal{D}(\mathfrak{h})^W.v_{\chi}^j.\]

Now, set $N = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} S(\mathfrak{h}^*)$ and $N_\chi = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \otimes_{A(\mathfrak{g})} V_{\chi}$. We have

\[(1.2) \quad N = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G)\]

and, using (1.1),

\[(1.3) \quad N = \bigoplus_{\chi \in W^*} N_\chi \otimes_{\mathbb{C}} V_{\chi}.\]

Then each $N_\chi$ is a simple (holonomic) $\mathcal{D}(\mathfrak{g})$-module [13] and, therefore, $N$ is a semisimple $\mathcal{D}(\mathfrak{g})$-module (see also [9]). Let $\mathcal{C}(N)$ be the full subcategory of finitely generated $\mathcal{D}(\mathfrak{g})$-modules of the form $\bigoplus_{\chi \in W^*} m_\chi N_\chi$, $m_\chi \in \mathbb{N}$. From [13] we know that the category $\mathcal{C}(N)$ is equivalent to the category $W$-mod (of finite dimensional $W$-modules) via the functor

\[\text{Sol} : \mathcal{C}(N) \longrightarrow W\text{-mod}, \quad \text{Sol}(N) = \text{Hom}_{\mathcal{D}(\mathfrak{g})^W}(N^G, S(\mathfrak{h}^*))\]

where $W$ acts on Sol$(N)$ through its natural action on $S(\mathfrak{h}^*)$.

The Killing form $\kappa$ induces a $G$-isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ and an algebra automorphism $\kappa$ of $\mathcal{D}(\mathfrak{g})$, defined by $\kappa(\partial(v)) = \kappa(v, \_)$, $\kappa(\kappa(v, \_)) = -\partial(v)$, for all $v \in \mathfrak{g}$. Hence, in coordinates, $\kappa(\partial_j) = x_j$, $\kappa(x_j) = -\partial_j$. Set $i = \sqrt{-1} \in \mathbb{C}$ and denote by $\mathcal{D}(\mathfrak{g})$ the automorphism of $\mathcal{D}(\mathfrak{g})$ given by $\mathcal{D}(\partial_j) = -i\partial_j$, $\mathcal{D}(x_j) = ix_j$. Define then the “Fourier transformation” $F_\mathfrak{g} \in \text{Aut}(\mathcal{D}(\mathfrak{g}))$ by $F_\mathfrak{g} = 1 \circ \kappa = \kappa \circ 1^{-1}$; thus $F_\mathfrak{g}(x_j) = ix_j$, $F_\mathfrak{g}(\partial_j) = i\partial_j$. One easily checks that $\kappa(\tau(x)) = F_\mathfrak{g}(\tau(x)) = \tau(x)$ for all $x \in \mathfrak{g}$; moreover, $\kappa$ and $F_\mathfrak{g}$ are $G$-equivariant. Similarly, since $\kappa$ is non degenerate and $W$-invariant on $\mathfrak{h}$, one can define $W$-equivariant automorphisms $\kappa$ and $F_\mathfrak{h} = 1 \circ \kappa$ in Aut $\mathcal{D}(\mathfrak{h})$.

**Lemma 1.2.** One has $\delta \circ F_\mathfrak{g} = F_\mathfrak{h} \circ \delta$.

*Proof.* A direct computation shows that $\delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P))$ when $P$ belongs to $S(\mathfrak{g})^G$ or $S(\mathfrak{g}^*)^G$. Since $\delta$ is a homomorphism, it follows that $\delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P))$ for all $P \in \mathbb{C}(S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G)$. Now, let $D \in \mathcal{D}(\mathfrak{g})^G$ and write $D = P + Q$ as in Lemma 1.1. Then, since $F_\mathfrak{g}(\mathcal{I}) = \mathcal{I}$, we have $\delta(F_\mathfrak{g}(D)) = \delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P)) = F_\mathfrak{h}(\delta(D))$. \hfill \Box
Recall that $H(h^*)$ is the vector space of $W$-harmonic polynomials on $h$. Hence

$$H(h^*) = \{ f \in S(h^*) : \partial(q).f = 0 \text{ for all } q \in S_+(h)^W \}$$

and, as $W$-module, $H(h^*)$ identifies with the regular representation of $W$. The vector space $H(h^*)$ is a graded subspace of $S(h^*)$ and we set $H^j(h^*) = S_j(h^*) \cap H(h^*)$, $0 \leq j \leq \nu$. Define the harmonic elements of $S(h)$ by $H(h) = F_h(H(h^*)) = \oplus_{j=1}^\nu H^j(h)$. (We could as well have set $H(h) = \chi(H(h^*))$, since $H^j(h^*)$ is stable under 1.)

Since $V_\chi \subset H^{\nu(\chi)}(h^*)$, we have $(E_h - b(\chi)).v_\chi^j = 0$. For all $d \in L := \text{ann}_{D(h^*)}^W(v_\chi^j)$, we have $[E_h - b(\chi), d] = [E_h, d] \in L$. It follows that $L = \bigoplus_{k \in \mathbb{L}} L \cap D_k(h^*)$, where $D_k(h^*) = \{ d \in D(h) : [E_h, d] = kd \}$. Equivalently, $L$ is stable under the $C^*$-action on $D(h)$ given by $f \mapsto \lambda f$, $\partial(v) \mapsto \lambda^{-1}\partial(v)$, $f \in h^*$, $v \in h$. In particular, we see that $F_h(L) = \chi(L)$.

Let $R$ be a ring and $\alpha \in \text{Aut}(R)$. If $M$ is an $R$-module, we define the $R$-module $M^\alpha$ to be the abelian group $M$ with action of $\alpha \in R$ on $x \in M$ given by $a.x = \alpha(a)x$. This applies to the modules $N$, $N_\chi$ and the automorphism $\alpha = F_h^{-1}$. Define

$$M = N^{F_h^{-1}}, \quad M_\chi = N_\chi^{F_h^{-1}}.$$ 

Thus, from (1.2) and (1.3), we obtain

$$M = D(g)/(D(g)\tau(g) + D(g)S_+(g)^G) \cong \bigoplus_{\chi \in W^*} M_\chi \otimes_C V_\chi.$$

Remark. In [13] one defines $M_\chi$ to be $N_\chi^{\tau^{-1}}$, but the two definitions agree. Indeed, let $V_\chi \cong D(h)^W.v_\chi^j = D(h)^W/L$ be as above. Then,

$$N_\chi = D(g)/J, \quad J = D(g)\tau(g) + D(g)S_+(g)^G + D(g)\delta^{-1}(L).$$

Write $N_\chi = D(g).(1 \otimes_{A(g)} v_\chi^j)$, where 1 is the canonical generator of $D(g)/D(g)\tau(g)$. From $\delta(E_h) = E_h - \nu$, we get that $(E_h - (b(\chi) - \nu))(1 \otimes_{A(g)} v_\chi^j) = 0$. It follows (as above) that $J$ is stable under the natural $C^*$-action on $D(g)$, Hence, $F_h(J) = \chi(J)$ and we have $N_\chi^{\tau^{-1}} = N_\chi^{F_h^{-1}}$.

We can define the category $C(M)$ similar to $C(N)$. We clearly have $M \in C(M)$ if, and only if, $N = M^{F_h} \in C(N)$. Moreover, by [13], this is equivalent to saying that $M$ is a $G$-equivariant finitely generated $D(g)$-module such that $M = D(g)M^G$ and $\text{Supp} M \subset N(g)$. This is also equivalent to: $N$ is a $G$-equivariant finitely generated $D(g)$-module such that $N = D(g)N^G$ and $N$ is $S_+$-finite (meaning that each $v \in N$ is killed by a power of $S_+(g)^G$).

Recall that $N_\chi^{F_h^{-1}} \cong V_\chi$ through the identification of $A(g)$ with $D(h)^W$.

**Lemma 1.3.** One has: $M_\chi^{F_h^{-1}} \cong V_\chi^{(V_\chi)^{F_h^{-1}}}$.

**Proof.** Write $N_\chi = D(g)/J$. Then, $M_\chi = D(g)/F_h(J)$ and $M_\chi^{F_h^{-1}} = D(g)^G/F_h(J^G)$. By Lemma 1.2, $\delta(F_h(J^G)) = F_h(\delta(J^G))$, therefore $M_\chi^{F_h^{-1}} \cong D(h)^W/F_h(\delta(J^G))$. Since $V_\chi \cong D(h)^W/\delta(J^G)$, the lemma follows. $\square$

Let $g_0$ be a real form of $g$ with adjoint group $G_0 \subset G$. There exists a natural action of $D(g)$ on $Db(g_0)$ defined by

$$(\partial(v).T,f) = \langle T,-\partial(v).f \rangle, \quad \langle \xi,T,f \rangle = \langle T,\xi.f \rangle$$
for all \( T \in \text{Db}(\mathfrak{g}_0), \, f \in C_c^\infty(\mathfrak{g}_0), \, v \in \mathfrak{g}, \, \xi \in \mathfrak{g}^* \). This induces a structure of \( \mathcal{D}(\mathfrak{g})^G \)-module on \( \text{Db}(\mathfrak{g}_0)^{G_0} \). From \( \mathcal{I}, \text{Db}(\mathfrak{g}_0)^{G_0} = 0 \), we obtain a natural \( A(\mathfrak{g}) \)-module structure on \( \text{Db}(\mathfrak{g}_0)^{G_0} \).

Fix a basis \( \{ u_1, \ldots, u_n \} \) of \( \mathfrak{g}_0 \) such that \( \kappa(u_j, u_k) = \pm \delta_{jk} \) and denote by \( dy \) be the Lebesgue measure associated to this choice. Let \( S(\mathfrak{g}_0) \) be the Schwartz space on \( \mathfrak{g}_0 \). Define, as in [18, Appendix 1], the Fourier transform of \( f \in S(\mathfrak{g}_0) \) by

\[
\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathfrak{g}_0} f(y) e^{-i\kappa(y, x)} \, dy
\]

Let \( T \) be a tempered distribution on \( \mathfrak{g}_0 \). The Fourier transform of \( T \) is defined by \( \langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle \) for \( f \in C_c^\infty(\mathfrak{g}_0) \). Then we have

\[
\forall \, D \in \mathcal{D}(\mathfrak{g}), \, T \in \text{Db}(\mathfrak{g}_0), \quad \hat{D} \hat{T} = F_{\hat{\mathfrak{g}}}(D).\hat{T}.
\]

Recall [2] that \( T \in \text{Db}(\mathfrak{g}_0) \) is said to be homogeneous of degree \( d \) if, for all \( f \in C_c^\infty(\mathfrak{g}_0), \, t \in \mathbb{R}^+, \, \langle T, f \rangle = t^d \langle T, f \rangle \), where \( f(t) = t^{-n} f(t^{-1} v) \). Then, a homogeneous distribution of degree \( d \) is tempered and satisfies \( E_{\mathfrak{g},T} = dT \). We will need the following well known result:

**Lemma 1.4.** Let \( T \in \text{Db}(\mathfrak{g}_0) \) be tempered and set \( M = \mathcal{D}(\mathfrak{g}) \cdot T \). Then \( M^{F_\mathfrak{g}} \cong \mathcal{D}(\mathfrak{g}) \cdot \hat{T} \).

**Proof.** By (1.4) we have \( \text{ann}_{\mathcal{D}(\mathfrak{g})}(\hat{T}) = F_{\hat{\mathfrak{g}}}^{-1}(\text{ann}_{\mathcal{D}(\mathfrak{g})}(T)) \). Hence the result. \( \square \)

Let \( N(\mathfrak{g}_0) \) be the set of nilpotent elements of \( \mathfrak{g}_0 \). Define \( \mathcal{D}(\mathfrak{g}) \)-submodules of \( \text{Db}(\mathfrak{g}_0) \) by

\[
\text{Db}(\mathfrak{g}_0)_{\text{nil}} = \{ \Theta \in \text{Db}(\mathfrak{g}_0) : \text{Supp} \Theta \subset N(\mathfrak{g}_0) \}
\]

\[
\text{Db}(\mathfrak{g}_0)_{S_+} = \{ T \in \text{Db}(\mathfrak{g}_0) : \exists k \in \mathbb{N}, \ (S_+(\mathfrak{g})^G)^k \cdot T = 0 \}.
\]

The elements of \( \text{Db}(\mathfrak{g}_0)_{S_+} \) are called \( S_+ \)-finite. Observe that \( \text{Db}(\mathfrak{g}_0)_{\text{nil}}^{G_0} \) and \( \text{Db}(\mathfrak{g}_0)_{S_+}^{G_0} \) are \( \mathcal{D}(\mathfrak{g})^G \)-modules. The next theorem is consequence of the results proved in [18].

**Theorem 1.5.** (1) \( \text{Db}(\mathfrak{g}_0)_{\text{nil}}^{G_0} = \{ \Theta \in \text{Db}(\mathfrak{g}_0)^{G_0} : \mathcal{D}(\mathfrak{g}) \cdot \Theta \subset C(M) \} \).

(2) \( \text{Db}(\mathfrak{g}_0)_{S_+}^{G_0} = \{ T \in \text{Db}(\mathfrak{g}_0)^{G_0} : \mathcal{D}(\mathfrak{g}) \cdot T \subset C(N) \} \).

(3) \( \Theta \in \text{Db}(\mathfrak{g}_0)_{\text{nil}}^{G_0} \iff \hat{\Theta} \in \text{Db}(\mathfrak{g}_0)_{S_+}^{G_0} \).

**Proof.** (1) follows from [18, Theorem 6.1], since \( \mathcal{D}(\mathfrak{g}) \cdot \Theta \subset C(M) \) is equivalent to \( \mathcal{D}(\mathfrak{g})^G \cdot \Theta \subset \bigoplus_{\chi \in \mathcal{W}} m_{\chi} \mathcal{M}_\chi^G \).

(2) and (3) are consequences of (1) and Lemma 1.4. \( \square \)

**Remark 1.6.** Let \( T \in \text{Db}(\mathfrak{g}_0)_{S_+}^{G_0} \). Recall that by the Harish-Chandra regularity theorem, \( T \) is given by

\[
\langle T, f \rangle = \int_{\mathfrak{g}_0^\prime} F_T(y)f(y) \, dy
\]

for some analytic function \( F_T \) on \( \mathfrak{g}_0^\prime \), locally integrable on \( \mathfrak{g}_0 \).
2. The distributions $\Theta_{u,\Gamma}$ and $T_{p,\Gamma}$

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, with adjoint group $G_0$, $\mathfrak{h}_0$ a Cartan subalgebra and let $H_0$ be the associated Cartan subgroup. Set $\mathfrak{h} = \mathbb{C} \otimes \mathfrak{h}_0$ and adopt the notation of §1. Denote by $W(h_0)$ the real Weyl group, i.e. $W(h_0) = N_{G_0}(h_0)/Z_{G_0}(h_0)$. Define
\[
\Delta_R = \{ \alpha \in \Delta : \alpha(h_0) \subset \mathbb{R} \} \quad \text{(the real roots)}
\]
\[
\Delta_I = \{ \alpha \in \Delta : \alpha(h_0) \subset i\mathbb{R} \} \quad \text{(the imaginary roots)}.
\]

A root which is neither real nor imaginary is called complex. Let $\Delta^+_I$ be a positive system of roots in $\Delta_I$ and set $\pi_I = \prod_{\alpha \in \Delta^+_I} \alpha$. Then each $w \in W(h_0)$ permutes the imaginary roots and one can define a character of $W(h_0)$, the imaginary signature, by
\[
\varepsilon_I : W(h_0) \rightarrow \{ \pm 1 \}, \quad w.\pi_I = \varepsilon_I(w)\pi_I.
\]

If $V$ is a $W(h_0)$-module we denote by $V^\varepsilon_I$ the isotypic component of type $\varepsilon_I$ in $V$.

In the sequel, we adopt the notation of [5] with the minor difference that we use $e^{-in(x,y)}$ in the definition of the Fourier transform.

Let $h \in \mathfrak{h}_0^*$ and $f \in C^\infty_c(\mathfrak{g}_0)$. Define [5, §3.1] the distribution $\mu_{G_0,h}$ by
\[
\langle \mu_{G_0,h}, f \rangle = |\det \text{ad}_{\mathfrak{g}_0/h_0}(h)|^\frac{1}{2} \int_{G_0/H_0} f(\hat{g},h)d\hat{g}
\]
Then one defines the function $J_{\mathfrak{g}_0}(f)$, or simply $J(f)$, on $\mathfrak{h}_0^*$ by
\[
J_{\mathfrak{g}_0}(f) = \{ h \mapsto \langle \mu_{G_0,h}, f \rangle \}.
\]

Set $\mathfrak{h}_0^{\text{reg}} = \{ h \in \mathfrak{h}_0 : \pi_I(h) \neq 0 \}$ and fix a connected component $\Gamma$ of $\mathfrak{h}_0^{\text{reg}}$. Let $u \in S(\mathfrak{h})$: Harish-Chandra has shown, see [17, §8.1, p. 123], that one can define a tempered $G_0$-invariant distribution on $\mathfrak{g}_0$ by
\[\tag{2.1}
\forall f \in C^\infty_c(\mathfrak{g}_0), \quad \langle \Theta_{u,\Gamma}, f \rangle = \lim_{h \rightarrow 0} \langle \partial(u), J(f) \rangle(h).
\]
Furthermore $\Theta_{u,\Gamma} \in \text{Db}(\mathfrak{g}_0)^G_+$ and, when $u \in S^b(\mathfrak{h})$, $\Theta_{u,\Gamma}$ is homogeneous of degree $-b - \mu - \ell$. Now let $p \in S(\mathfrak{h}^*)$ and define $T \in \text{Db}(\mathfrak{g}_0)^G_+$ by
\[\tag{2.2}
T_{p,\Gamma} = \Theta_{F_h(p),\Gamma} = \{ f \mapsto \lim_{h \rightarrow 0} \langle \partial(F_h(p)), J(f) \rangle(h) \}.
\]
Then, $T_{p,\Gamma}$ is tempered and is homogeneous of degree $b - \nu$ when $p \in S^b(\mathfrak{h}^*)$.

**Lemma 2.1.** (1) Let $\varphi \in S(\mathfrak{g}^*)^G$. Then, $\varphi T_{p,\Gamma} = T_{\delta(\varphi)p,\Gamma}$.

(2) Let $q \in S(\mathfrak{g})^G$. Then, $\partial(q).T_{p,\Gamma} = T_{\partial(\delta(q)),p,\Gamma}$.

**Proof.** Set $u = F_h(p)$, $\phi = \delta(\varphi) \in S(\mathfrak{h}^*)^W$ and $s = \delta(q) \in S(\mathfrak{h})^W$. Let $f \in C^\infty_c(\mathfrak{g}_0)$.

(1) By definition, see (2.2), $\langle \varphi T_{p,\Gamma}, f \rangle = \lim_{h \rightarrow 0} \langle \partial(u), J(\varphi f) \rangle(h)$. But, [17, Lemma 3.2.7, p. 38], (1.4) and Lemma 1.2 imply that $J(\tilde{\varphi f}) = \partial(F_h(\phi)).J(\tilde{f})$.

Hence,
\[
\langle \varphi T_{p,\Gamma}, f \rangle = \lim_{h \rightarrow 0} \langle \partial(u), \partial(F_h(\phi)).J(\tilde{f}) \rangle(h) = \lim_{h \rightarrow 0} \langle \partial(F_h(\phi p)), J(\tilde{f}) \rangle(h)
\]
\[
= \langle T_{\partial p,\Gamma}, f \rangle,
\]
as desired.
(2) By (1.4), \( \partial(q).T_{p,\Gamma} = \text{the Fourier transform of } F_b^{-1}(q)\Theta_{u,\Gamma} \), hence
\[
\langle \partial(q).T_{p,\Gamma}, f \rangle = \lim_{h \rightarrow 0} \langle \partial(u).J(F_b^{-1}(q)f) \rangle(h).
\]

Set \( g = J(f) \). From [17, Lemma 3.2.7, p. 38] and Lemma 1.2 we obtain that
\[
J(F_b^{-1}(q)f) = F_b^{-1}(s)g.
\]

Therefore
\[
\langle \partial(q).T_{p,\Gamma}, f \rangle = \lim_{h \rightarrow 0} \langle \partial(u).(F_b^{-1}(s)g) \rangle(h).
\]

Recall (see §1) that we have chosen a coordinate system \( \{x_j; e_j\}_{1 \leq j \leq \ell} \). With standard notation, we write \( x^\alpha = \prod_{k=1}^{\ell} x_k^{a_k}, e^\mu = \prod_{k=1}^{\ell} e_k^{\mu_k} \) and
\[
p = \sum_{a \in \mathbb{N}'^\ell} p_a x^a, \quad s = \sum_{\mu \in \mathbb{N}^\ell} s_\mu e^\mu.
\]

Set \( \partial^\alpha = \prod_j \partial(e_j)^{\alpha_j} \) and \( \partial = \sum_{\mu \in \mathbb{N}^\ell} s_\mu \partial^\mu \). Order \( \mathbb{N}^\ell \) by saying that \( \mu \leq \alpha \) if \( \mu_j \leq \alpha_j \) for all \( j \). Set \( \alpha! = \prod_j \alpha_j! \) and \( \alpha = \left( \begin{array}{c} \alpha \\ \mu \end{array} \right) = \prod_j \frac{\alpha_j!}{\mu_j!} \), when \( \mu \leq \alpha \). Then:
\[
\partial^\mu(x^\alpha) = \begin{cases} 0 & \text{if } \mu \not\leq \alpha, \\ \frac{\alpha!}{(\alpha-\mu)!} x^\alpha - \mu & \text{if } \mu \leq \alpha. 
\end{cases}
\]

Now we have \( u = F_b(p) = \sum_{a \in \mathbb{N}'^\ell} p_a i^{\alpha_a} \partial^\alpha \) and \( F_b^{-1}(s) = \sum_{\mu \in \mathbb{N}^\ell} g_\mu i^{-|\mu|} e^\mu \). Therefore, using the Leibniz formula, we get that
\[
\langle \partial(u).(F_b^{-1}(s)g) \rangle(h) = \sum_{\alpha} \sum_{\mu \leq \alpha} \beta \sum_{\mu \leq \alpha} p_\alpha s_\mu |\mu|! \beta! \lim_{h \rightarrow 0} \langle \partial^{\beta}(x^\alpha) \partial^\alpha - \beta \rangle(h).
\]

But \( \lim_{h \rightarrow 0} \partial^\beta(x^\alpha)\rangle(h) = 0 \) unless \( \beta = \mu \), hence
\[
\lim_{h \rightarrow 0} \langle \partial(u).(F_b^{-1}(s)g) \rangle(h) = \sum_{\alpha} \sum_{\mu \leq \alpha} p_\alpha s_\mu |\mu|! \beta! \lim_{h \rightarrow 0} \langle \partial^{\beta}(x^\alpha) \rangle(h).
\]

On the other hand, we have
\[
\langle T_{\partial(u)}(s), p, g \rangle = \lim_{h \rightarrow 0} \langle \partial(F_b(\partial(u)p)) \rangle(h).
\]

Since \( \partial(s) = \sum_{\alpha} \sum_{\mu \leq \alpha} \frac{\alpha!}{(\alpha-\mu)!} s_\mu p_a x^\alpha - \mu \), we obtain that
\[
\langle T_{\partial(u)}(s), p, g \rangle = \sum_{\alpha} \sum_{\mu \leq \alpha} \frac{1}{(\alpha-\mu)!} s_\mu p_a |\mu|! \lim_{h \rightarrow 0} \langle \partial^{\mu}(x^\alpha) \rangle(h).
\]

This proves the desired equality. □

**Theorem 2.2.** Let \( p \in S(h^*) \) and \( D \in \mathcal{D}(g) \). Then, \( D.T_{p,\Gamma} = T_{\partial(D)}(p,\Gamma) \).

**Proof.** Since \( T_{p,\Gamma} \) is \( G_0 \)-invariant, we have \( \mathcal{L}.T_{p,\Gamma} = 0 \). Let \( P \in \mathcal{C}(S(g)^G, S(g^*)^G) \); by Lemma 2.1 and an obvious induction, we obtain that \( P.T_{p,\Gamma} = T_{\partial(P)}(p,\Gamma) \). The theorem then follows from Lemma 1.1.

Recall, see Remark 1.6, that \( \tilde{\Theta}_{u,\Gamma} \in \mathcal{Db}(\mathfrak{g}_0)^{G_{\mathcal{A}}} \) is determined by a locally integrable function on \( \mathfrak{g}_0 \). We still denote this function by \( \tilde{\Theta}_{u,\Gamma} \).

**Lemma 2.3.** ([5, Lemma 6.1.2]) There exists \( c_P \in \mathbb{C}^* \), such that
\[
a_{\xi_j^+}(h)|det ad_{\mathfrak{h}_0}/\mathfrak{h}_0(h)|^{\frac{1}{2}} \tilde{\Theta}_{F_b(p),(\Gamma)}(h) = c_P(p)(h)
\]
for all \( p \in S(h^*)^G \) and \( h \in \mathfrak{h}_0^{reg} \). □
Remark. In the notation of the lemma, if $u = F_h(p)$, the function $\tilde{u}(ih)$ of [5] is replaced here by $p(h)$ since we are using $e^{-\langle x, y \rangle}$ in the definition of the Fourier transform.

**Theorem 2.4.** Let $p \in S(\mathfrak{h}^+)$. There exists a bijective map

$$\rho: \mathcal{D}(\mathfrak{g})^G \cdot T_{p,\Gamma} \rightarrow \mathcal{D}(\mathfrak{h})^W \cdot p, \quad \rho(D.T_{p,\Gamma}) = \delta(D).p$$

which, through $\delta$, yields an isomorphism

$$\rho: A(\mathfrak{g}).T_{p,\Gamma} \cong \mathcal{D}(\mathfrak{h})^W \cdot p$$

**Proof.** We first need to show that $\rho$ is well defined. Let $D \in \mathcal{D}(\mathfrak{g})^G$; by Theorem 2.2 we have

$$(\dagger) \quad D.T_{p,\Gamma} = T_{\delta(D),p,\Gamma} = \hat{\Theta}_{F_b(\delta(D),p,\Gamma)}.$$

Suppose that $D.T_{p,\Gamma} = 0$. Then, the analytic function associated to $T_{\delta(D),p,\Gamma} \in \text{Db}(\mathfrak{g})^{G^0}_{\delta}$ vanishes on $\mathfrak{h}_0^{\text{reg}}$. Notice that, since $\delta(D)$ is $W$-invariant, $\delta(D).p \in S(\mathfrak{h}^+)$. Therefore Lemma 2.3 gives $\delta(D).p = 0$ on $\mathfrak{h}_0^{\text{reg}}$. Thus $\delta(D).p = 0$ on $\mathfrak{h}$ and $\rho$ is well defined.

Now, it follows easily from $(\dagger)$ that $\rho$ is a linear bijection. Since $I.T_{p,\Gamma} = 0$, the last assertion is clear.

Recall that we denote by $V_{\chi} \subset H^{k(\chi)}(\mathfrak{h}^+)$ a simple $W$-module in the class of $\chi \in W^{-}$.

**Corollary 2.5.** Let $p \in S(\mathfrak{h}^+)$. Then there exists $\chi \in W^{-}$ such that $CW.p$ is simple. Then there exists $\chi \in W^{-}$ such that $V_{\chi}^{\text{reg}} \neq 0$. We have:

1. $\mathcal{D}(\mathfrak{g}).T_{p,\Gamma} \cong N_{\chi}$ and $\mathcal{D}(\mathfrak{g})^G \cdot T_{p,\Gamma} \cong V_{\chi}$;
2. $\mathcal{D}(\mathfrak{g}).\Theta_{F_b(p),\Gamma} \cong M_{\chi}$ and $\mathcal{D}(\mathfrak{g})^G \cdot \Theta_{F_b(p),\Gamma} \cong (V_{\chi})^{\delta^{-1}}$.

**Proof.** The first assertion follows from $H(\mathfrak{h}^+) \cong CW$. Then, 1 and 2 are consequences of $V_{\chi} \cong H(\mathfrak{h})^W \cdot p$, Lemma 1.3 and Theorem 2.4.

**Remark 2.6.** Let $\chi \in W^{-}$ be such that $V_{\chi}^{\text{reg}} \neq 0$. It follows obviously from the previous corollary that

$$N_{\chi} \cong \mathcal{D}(\mathfrak{g}).T_{p,\Gamma}, \quad M_{\chi} \cong \mathcal{D}(\mathfrak{g}).\Theta_{u,\Gamma},$$

where $0 \neq p \in V_{\chi}^{\text{reg}} \subset H^{k(\chi)}(\mathfrak{h}^+)$ and $u = F_{\delta}(p) \in H^{k(\chi)}(\mathfrak{h})$.

3. THE DECOMPOSITION OF $\text{Db}(\mathfrak{g}_0)_{G_{\mathcal{S}}}^{G_0}$ AND $\text{Db}(\mathfrak{g}_0)_{G_{\mathcal{U}}}^{G_0}$

Fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and let $[\mathfrak{h}_1], \ldots, [\mathfrak{h}_r]$ be the conjugacy classes of Cartan subalgebras in $\mathfrak{g}_0$. For each $j = 1, \ldots, r$ we denote by

$$\mathfrak{h}_{j,\mathcal{C}} = \mathfrak{h}_j \otimes \mathbb{C}, \quad W_j = W(\mathfrak{g}, \mathfrak{h}_{j,\mathcal{C}}), \quad \Delta_{I,j}^{+} \text{ a set of positive imaginary roots}, \quad \mathfrak{e}_{I,j} : W(h_j) = W(G_0, \mathfrak{h}_j) \rightarrow \{\pm 1\} \text{ the imaginary signature associated to } \mathfrak{h}_j.$$

For each $j$ we fix a connected component $\Gamma_j$ of $\mathfrak{h}_j^{\text{reg}}$. The results of §2 then apply to $\mathfrak{h}_0 = \mathfrak{h}_j$, $\Gamma = \Gamma_j$ etc.
Remark 3.1. Recall that the $h_{j,C}$ are $G$-conjugate. Therefore, if $1 \leq j, k \leq r$, the algebras $D(h_{j,C})^{W_j}$ and $D(h_{k,C})^{W_k}$ are naturally isomorphic. Denote this isomorphism by $\gamma_{jk}$ and let $\delta_j$ be the Harish-Chandra isomorphism from $A(g)$ onto $D(h_{j,C})^{W_j}$. One can check that $\delta_k = \gamma_{jk} \circ \delta_j$. Therefore, we can choose an "abstract" Cartan subalgebra $h$ and identify $\delta_j$ with the homomorphism $\delta : D(g)^G \rightarrow D(h)^W$, where $W = W(G, h)$. Then, if $\chi \in W^-_i$ we have an irreducible $W$-module $V_\chi \subset H^k(\chi)(h^*)$ and a simple $D(h)^W$-module $V_\chi$.

For each $\chi \in W_-$ choose a simple $W$-module $V_{\chi,j} \subset H^k(\chi)(h_{j,C})$, $V_{\chi,j} \cong V_\chi$. Write $V_{\chi,j} = V_{\chi,j}^{\varepsilon_l,j} \oplus E_{\chi,j}$ with $E_{\chi,j}$ stable under $W(h_j)$. Let $\{v_{\chi,j}^k\}_{1 \leq k \leq d(\chi)}$ be a basis of $V_{\chi,j}$ such that

$$V_{\chi,j}^{\varepsilon_l,j} = \bigoplus_{k=1}^{n_l(\chi)} C v_{\chi,j}^k, \quad E_{\chi,j} = \bigoplus_{k=n_l(\chi)+1}^{d(\chi)} C v_{\chi,j}^k$$

(hence $n_l(\chi) = \dim V_{\chi,j}^{\varepsilon_l,j}$).

Lemma 3.2. The $D(h_{j,C})^{W_j}$-module $S(h_{j,C}^*)^{\varepsilon_l,j}$ decomposes as

$$S(h_{j,C}^*)^{\varepsilon_l,j} = \bigoplus_{\chi \in \mathcal{W}} \bigoplus_{k=1}^{n_l(\chi)} D(h_{j,C})^{W_j} v_{\chi,j}^k$$

with $D(h_{j,C})^{W_j} v_{\chi,j}^k \cong V_\chi$.

Proof. Clearly, we can drop the index $j$ and write $h_0 = h_j$, $h = h_{j,C}$, $v_\chi^k = v_\chi^{k,j}$ etc. Since $D(h)^W v_\chi^k \subset S(h^*)^{\varepsilon_l}$ for $1 \leq k \leq d(\chi) = \dim V_{\chi,j}^{\varepsilon_l,j}$, one has

$$S(h^*)^{\varepsilon_l} \supset \bigoplus_{\chi \in \mathcal{W}} \bigoplus_{k=1}^{n_l(\chi)} D(h)^W v_\chi^k.$$ 

Recall from §1 that $S(h^*) = \bigoplus_{\chi} S(h^*)[\chi]$ with $S(h^*)[\chi] = \bigoplus_{k=1}^{d(\chi)} D(h)^W v_\chi^k$. Write $S(h^*)[\chi] = E_1 \oplus E_2$, where $E_1 = \bigoplus_{k=1}^{d(\chi)} D(h)^W v_\chi^k$ and $E_2 = \bigoplus_{k=n(\chi)+1}^{d(\chi)} D(h)^W v_\chi^k$.

Notice that $E_1, E_2$ are stable under $W(h_0)$ and that we have $S(h^*)[\chi]^{\varepsilon_l} = E_1 \oplus E_2^{\varepsilon_l}$.

We now show that $E_2^{\varepsilon_l} = 0$. This will prove that

$$S(h^*)^{\varepsilon_l} = \bigoplus_{\chi \in \mathcal{W}} \bigoplus_{k=1}^{n_l(\chi)} D(h)^W v_\chi^k.$$ 

Let $D \in D(h)^W$ and $v \in V_\chi$. Notice first that if $D.v \neq 0$, the operator $D$ yields an isomorphism of $W$-modules $V_\chi \cong D.V_\chi$. Therefore, if $V_\chi = \bigoplus_{k} S_k$ with $S_k$ irreducible $W(h_0)$-module, we get that $D.V_\chi = \bigoplus_k D.S_k$, $D.S_k \cong S_k$. It follows that if $v \in E_1$ (the $(W(h_0))$-stable complement of $V_{\chi,j}^{\varepsilon_l,j}$) then $D.v \in D.E_1$ with $D.E_1 \cap S(h^*)^{\varepsilon_l} = 0$. Let $p = \sum_{k=n(\chi)+1}^{d(\chi)} D_k.v_\chi^k \in E_2$. Then, $CW(h_0).p \subset \sum_{k=n(\chi)+1}^{d(\chi)} CW(h_0).D_k.v_\chi^k$ and, by the previous remarks, $(CW(h_0).D_k.v_\chi^k)^{\varepsilon_l} = 0$. Thus $(CW(h_0).p)^{\varepsilon_l} = 0$, which shows that $E_2^{\varepsilon_l} = 0$. \hfill $\square$

Recall the following result:

Proposition 3.3. ([5, Proposition 6.1.1]) (1) The linear map $T : \bigoplus_{j=1}^r S(h_{j,C}^*)^{\varepsilon_l,j} \rightarrow Db(g_0)^G_{S_+}$, $T(p_1, \ldots, p_r) = \sum_{j=1}^r T_{p_j, r_j}$ is an isomorphism of vector spaces.

(2) The map $T$ induces an isomorphism:

$$\bigoplus_{j=1}^r H(h_{j,C}^*)^{\varepsilon_l,j} \cong \{ T \in Db(g_0)^G_{S_+} : S_+(g)^G.T = 0 \}.$$ 

Proof. (2) follows from the proof of [5, Proposition 6.1.1]. \hfill $\square$
Theorem 3.4. Set $T(h_j) = \sum_{p \in S(h_j, c)^f r, j} c T_{p, r_j}$. Then we have the following decomposition of $D(g)^G$-modules:

$$Db(g_0)^{G_0}_{S_k} = \bigoplus_{j=1}^r T(h_j)$$

with

$$T(h_j) = \bigoplus_{\chi \in W^-} \bigoplus_{k=1}^r D(g)^G.T_{\chi, j, r_j}$$

and $D(g)^G.T_{\chi, j, r_j} \cong N^G_{\chi}$. 

Proof. The decomposition of $T(h_j)$, as a $D(g)^G$-module, is consequence of Theorem 2.4, Lemma 3.2 (using the isomorphism $\delta_j : A(g) \cong D(h_j, c)^W$) and Proposition 3.3. The decomposition of $Db(g_0)^{G_0}_{S_k}$ follows from Proposition 3.3. \qed

Using the Fourier transform, we obtain the following:

Corollary 3.5. The $D(g)^G$-module $Db(g_0)_{nil}^{G_0}$ decomposes as

$$Db(g_0)_{nil}^{G_0} = \bigoplus_{r=1}^r \bigoplus_{\chi \in W^-} \bigoplus_{k=1}^r D(g)^G.\Theta^{-1}(\chi, r, \Gamma_j)$$

with $D(g)^G.\Theta^{-1}(\chi, r, \Gamma_j) \cong M^G_\chi$. \qed

The next corollary follows from Theorem 3.4 and Corollary 3.5.

Corollary 3.6. We have:

$$Db(g_0)^{G_0}_{S_k} \cong \bigoplus_{\chi \in W^-} N^G_{\chi}$$

$$Db(g_0)_{nil}^{G_0} \cong \bigoplus_{\chi \in W^-} M^G_{\chi}$$

where $m_{\chi} = \sum_{r=1}^r \dim V_{\chi, j, r}$. \qed

Remark 3.7. Let $\chi \in W^-$. It is not always possible to "realize" the modules $N_{\chi}$ and $M_{\chi}$ as $D(g).T$ for some $T \in Db(g_0)$, where $g_0$ is a real form of $g$. By the previous results, this statement is equivalent to the existence of a Cartan subalgebra $\mathfrak{h}_j \subset g_0$ such that $V_{\chi, r, j} \neq 0$. D. Renard has observed that, using the results of W. Rossmann [15], this can be translated to a question about centralizers of nilpotent elements. Fix a real form $g_R$ of $g$ with adjoint group $G_R$. If $x \in g_R$ is nilpotent one defines a subgroup of the component group $A(G, x)$ (see §4 for notation) by

$$A(G, x) = G_R^g/G_R \cap (G^x)^0.$$ 

Recall that $\chi \in W^-$ can be written $\sigma(\mathfrak{O}, \psi)$ via the Springer correspondence, where $\mathfrak{O} \subset g$ is a nilpotent orbit and $\psi : A(\mathfrak{O}) \to GL(E)$ is an irreducible representation. Then, by [15, Corollary 3.2 & Theorem 3.3], there exists a Cartan subalgebra $h_0 \subset g_R$ such that $V_{\chi, r, j} \neq 0$ if, and only if, there exists a nilpotent element $\varepsilon \in g_R$ such that $\mathfrak{O} = G.x$ and $E^A(G, x) \neq 0$.

Let $g = \mathfrak{sp}(\ell, \mathbb{C})$ and let $\phi \in W^-$ be the long sign character, i.e. $V_\phi = \mathbb{C}_{\pi \ell}$ where $\pi_{\ell}$ is the product of the long roots. Then, see [6, §13.3], $\phi = \sigma(\mathfrak{O}, \psi)$ where $\mathfrak{O} = G.x$ is the subregular nilpotent orbit with partition $[2\ell - 2, 2]$ and $\psi$ is the non trivial character of $A(\mathfrak{O}) \cong \{ \pm 1 \}$. The real forms of $g$ are $\mathfrak{sp}(\ell, \mathbb{R})$ and the $\mathfrak{sp}(p, q)$, $p+q = \ell$. Assume now that $\ell \geq 3$. By the classification of nilpotent orbits in $\mathfrak{sp}(p, q)$, see [7, Theorem 9.2.5], we know that $O^-\mathfrak{sp}(p, q) = 0$. Hence, by Rossmann's results, $V_{\varepsilon, r, j} = 0$ for each Cartan subalgebra $\mathfrak{h}_j \subset \mathfrak{sp}(p, q)$. On the other hand, if $G_R$ is the adjoint group of $\mathfrak{sp}(\ell, \mathbb{R})$, one can show that $A(G_R, x) = A(G, x)$. Thus, with the above notation, $E^A(G_R, x) = 0$ and it follows that $V_{\varepsilon, r, j} = 0$ for each Cartan subalgebra $\mathfrak{h}_j \subset \mathfrak{sp}(\ell, \mathbb{R})$. For instance, when $g = \mathfrak{sp}(3, \mathbb{R})$ there are six conjugacy
classes of Cartan subalgebras and one can directly verify (without using [15]) that $V_{\phi}^{\varepsilon_j} = 0$ for $j = 1, \ldots, 6$. We thank D. Renard for showing this computation to us.

Let $x \in N(g_0)$ and denote by $\beta_x$ the Liouville (Kostant-Kirillov) measure on $G_0.x$. By [14] one can define $\Theta_x \in Db(g_0)^{G_0}_{nil}$ by $(\Theta_x)_f = \int_{G_0.x} f d\beta_x$ for all $f \in C^\infty_c(g_0)$. Set $O = G.x$. Then, see [9], [10] or [18], $\Theta_x$ is homogeneous of degree $\lambda_O = 1/2 \dim O - \dim g$ and satisfies

$$D(g) \Theta_x \cong M_{\chi_O}$$

for some $\chi_O \in W^-$ such that $\lambda_O = \nu - n - b(\chi_O)$.

**Corollary 3.8.** There exists $j \in \{1, \ldots, r\}$ and $u \in F_{\nu}^{-1}(V_{\chi_O})^{\varepsilon_j}$ such that

$$D(g)^G \Theta_x \cong D(g)^G \Theta_u R_j.$$

**Proof.** Since $D(g)^G \Theta_x \cong M_{G \chi_O}^G$ is a simple submodule of $Db(g_0)^{G_0}_{nil}$, the claim follows from Corollary 3.5.

**Remark 3.9.** It is proved in [1], see also [5], that $\Theta_x$ can be written as $\sum_{j=1}^r \Theta a_j R_j$ with $a_j \in H^{\varepsilon(x)}(G, C)^{\varepsilon_j}$. It is easily seen that we may assume $GW a_j \cong V_{\chi_O}$ for all $j$ such that $a_j \neq 0$. W. Rossmann [15] has given conditions to ensure that $\Theta_x = \Theta a_j R_j$ for some $j$.

4. **Example: the complex case**

We assume in this section that $g_0 = g_1^R$ is a complex semisimple Lie algebra, $g_1$, viewed as a real Lie algebra. Then, $g$ can be identified with $g_1 \times g_1$ and $g_0$ with the diagonal $\{(a, a) : a \in g_1\}$. Let $h_1$ be a Cartan subalgebra of $g_1$. Recall the following well known facts, see [17] or [18]:

- $h_0 = \{(a, a) : a \in h_1\}$ is a Cartan subalgebra of $h_0$ and $h = h_0 \oplus R \mathbb{C} = h_1 \times h_1$;
- $W(h, h) = W_1 \times W_1$, where $W_1 = W(g_1, h_1)$, and $W(h_0) = \{(w, w) : w \in W\}$ is isomorphic to $W_1$;
- there is a unique conjugacy class $[h_0]$ of Cartan subalgebras and $h_0'$ is connected;
- the roots in $\Delta(g, h)$ are complex and, therefore, $\varepsilon_j = 1$;
- the irreducible representations of $W$ are of the form $\chi = \phi \otimes \mu$, $\phi, \mu \in W_1$;
- one has $\phi = \phi^*$ for all $\phi \in W_1^\ast$, where $\phi^*$ is the dual representation.

Observe that $D(g) = D(g_1) \otimes D(g_1)$ and $D(g)^G = D(g_1)^G \otimes D(g_1)^G$.

**Lemma 4.1.** Let $\chi \in W^-$. Then, the simple $D(g)$-module $M_\chi$ is of the form $M_\phi \otimes M_\mu$ for some $\phi, \mu \in W_1^\ast$.

**Proof.** The claim follows easily from the definition of the category $C(M)$ and the decomposition of the $W$-module $S(h^\ast) = S(h_1^\ast) \otimes S(h_1^\ast)$.

**Corollary 4.2.** ([18, Theorem 6.11]) We have

$$Db(g_0)^{G_0}_{nil} \cong \bigoplus_{\phi \in W_1^\ast} M^{G_1}_\phi \otimes M^{G_1}_\phi$$

as a $D(g)^G$-module.

**Proof.** Let $\chi = \phi \otimes \mu \in W^-$. Then, $V_{\chi}^{\varepsilon_j} = (V_{\phi} \otimes V_{\mu})^{W_1} \neq 0$ if, and only if, $\phi = \mu$ and therefore $n(\chi) = 1$. The assertion now follows from Corollary 3.5.
Recall the following general results from [13]. Since the module $M_\chi$ is irreducible and $G$-equivariant, its support is the closure of a nilpotent orbit $O = G.x$. Furthermore, if $\iota : O \hookrightarrow g$ is the inclusion, $M_\chi$ is uniquely determined by its ($D$-module) inverse image $\mathcal{L}_\chi := \iota^*M_\chi$. The $D_O$-module $\mathcal{L}_\chi$ is an irreducible integrable connection associated to an irreducible representation $\psi$ of the component group $A(O) := G^\circ/(G^\circ)^0$ (where $(G^\circ)^0$ is the connected component of the centralizer $G^\circ$). Therefore, since $\chi$ is uniquely determined by $O$ and $\psi$, we set $\chi = \sigma(O, \psi)$.

In our situation, i.e. in the complex case, we have $O = O_1^j \times O_1^j$ with $O_1^j$ nilpotent orbits in $g_1$ for $j = 1, 2$. Then, $\chi = \sigma(O, \psi) = \phi_1 \boxtimes \phi_2$, $\mathcal{L}_\chi = \mathcal{L}_{\phi_1} \boxtimes \mathcal{L}_{\phi_2}$, $\phi_j = \sigma(O_1^j, \psi_j)$, $\psi = \psi_1 \boxtimes \psi_2$. Note that $b(\chi) = b(\phi_1) + b(\phi_2)$ and $\lambda_O = \lambda_{O_1^j} + \lambda_{O_2^j}$.

Let $x \in N(g_0)$; set $x = (x_1, x_1)$, $x_1 \in N(g_1)$, $O_1 = G_1.x_1$, $O = G.x = O_1 \times O_1$. The inclusion $\iota : O \hookrightarrow g$ is equal to $\iota_1 \times \iota_1$, where $\iota_1 : O_1 \hookrightarrow g_1$. By (3.1) and Corollary 4.2 there exist $\chi \in W^-, \chi_1 \in W_1^-$ such that $\chi = \chi_1 \boxtimes \chi_1$ and $D(g).\Theta_\chi \cong M_{\chi_1} \boxtimes M_{\chi_1}$.

It is known (Harish-Chandra) that $\Theta_{X_1} = \Theta_{h, b'_0}$ for some $u \in S(h_1) \boxtimes S(h_1)$. The following result has been proved by various authors; see [2, 3] (when $O_1$ is “special”), [8], [9], [16].

**Theorem 4.3.** One has $\chi_1 = \sigma(O_1, \text{triv})$, and there exists $p \in (V_{\chi_1} \boxtimes V_{\chi_1})^W_1$ such that $\Theta_{X_1} = \Theta_{F(p), b'_0}$.

**Proof.** Recall from [9] or [10] that $\chi = \chi_1 \boxtimes \chi_1 = \sigma(O, \text{triv})$. This means that

$$\mathcal{L}_\chi = \mathcal{L}_{\chi_1} \boxtimes \mathcal{L}_{\chi_1} = O_O = O_{O_1} \boxtimes O_{O_1},$$

(where we denote by $O_X$ the structural sheaf of an algebraic variety $X$). This yields $\mathcal{L}_{\chi_1} = O_{O_1}$ and $\chi_1 = \sigma(O_1, \text{triv})$.

Set $T_x = \Theta_{X_1}$; then $D(g)^T_x = N_{\chi_1} \boxtimes N_{\chi_1}$ (see Lemma 1.4). Since $S_+(g^*)G.\Theta_x = 0$ we have $S_+(g^*)G.\Theta_x = 0$. It follows from Proposition 3.3(2) that we can write $T_x = T_{p, b'_0}$ for some $p \in (H(h_1^*) \boxtimes H(h_1^*))^{W_1}$ or equivalently, $\Theta_x = \Theta_{F(p), b'_0}$. Now, by Theorem 2.4, $D(h)^W.p = V_{\chi_1} \boxtimes V_{\chi_1}$ and therefore $CW.p \cong V_{\chi_1} \boxtimes V_{\chi_1}$. Moreover, $T_x = T_{p, b'_0}$ is homogeneous of degree $b(\chi_0) - 2\nu = 2b(\chi_1) - 2\nu = \deg p - 2\nu$. Thus $\deg p = 2b(\chi_1)$ and, by definition of $V_{\chi_1}, p \in (V_{\chi_1} \boxtimes V_{\chi_1})^{W_1}$.

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