When is a complex elliptic curve the product of two real algebraic curves?

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1 Introduction and the main results

It is well known that a complex elliptic curve \( E \), considered as a real Lie group, is isomorphic to the product \( S^1 \times S^1 \) of two circles (cf. [4]). However, such a statement is not fully satisfactory, because it says nothing about the underlying real algebraic structure \( E_{\mathbb{R}} \) of \( E \). Recall that a complex projective variety \( V \subseteq \mathbb{P}^n(\mathbb{C}) \) can be, in the obvious way, considered as a real algebraic variety and, as such, will be denoted by \( V_{\mathbb{R}} \). Of course, \( \dim V_{\mathbb{R}} = 2 \dim V \).

Moreover, \( V_{\mathbb{R}} \) is an affine real algebraic variety (cf. [1] Proposition 3.4.8). (For the background material on real algebraic geometry, the reader may refer to the book [1]. By an affine real algebraic variety we mean a locally ringed space, isomorphic to an algebraic subset of \( \mathbb{R}^n \), for some \( n \), equipped with the sheaf of \( \mathbb{R} \)-valued regular functions; cf. [1] Chapter 3).

The underlying real algebraic surface \( E_{\mathbb{R}} \) of a complex elliptic curve \( E \) is definitely not biregularly isomorphic to \( S^1 \times S^1 \). (Clearly, every regular mapping from \( S^1 \times S^1 \) into \( E_{\mathbb{R}} \) is constant. Indeed, the existence of a nonconstant real regular mapping \( \mathbb{P}^1(\mathbb{R}) \cong S^1 \rightarrow E_{\mathbb{R}} \) would imply the existence of a nonconstant complex regular mapping from \( \mathbb{P}^1(\mathbb{C}) \) into \( E \), which is impossible; see Section 2). The aim of this paper is to answer the question: When is \( E_{\mathbb{R}} \) biregularly isomorphic to the product of two real algebraic curves?
Before stating our results we need some preparation. Recall that a complex elliptic curve $E$ is said to have complex multiplication if its ring of endomorphisms $\text{End}(E)$ is isomorphic to an order in an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, that is, $\text{End}(E) \cong \mathbb{Z} + f\mathcal{O}_d$, for some strictly positive integers $f, d$, with $d$ square free, where $\mathcal{O}_d$ is the ring of integers in the field $\mathbb{Q}(\sqrt{-d})$. The number

$$
\delta(E) = \begin{cases} 
-f^2d & \text{for } d \equiv 3 \mod 4 \\
-4f^2d & \text{for } d \equiv 1, 2 \mod 4
\end{cases}
$$

is called the discriminant of $\text{End}(E)$.

Let $H_1^{\text{alg}}(E_{\mathbb{R}}, \mathbb{Z}/2)$ be the subgroup of the homology group $H_1(E_{\mathbb{R}}, \mathbb{Z}/2)$ which consists of all homology classes represented by real algebraic curves contained in $E_{\mathbb{R}}$ (cf. [1] Chapter 11).

Our main results are contained in the following three theorems.

**Theorem 1** Given a complex elliptic curve $E$, the following conditions are equivalent:

(i) The underlying real algebraic surface $E_{\mathbb{R}}$ of $E$ is biregularly isomorphic to the product of two real algebraic curves.

(ii) $E$ has complex multiplication and the discriminant $\delta(E)$ of $\text{End}(E)$ is odd.

(iii) $H_1^{\text{alg}}(E_{\mathbb{R}}, \mathbb{Z}/2) = H_1(E_{\mathbb{R}}, \mathbb{Z}/2)$.

Clearly, (i) $\Rightarrow$ (iii) is trivial, while (iii) $\Rightarrow$ (ii) is shown in [2]. We shall prove (ii) $\Rightarrow$ (i) in Section 3.

The variety $E_{\mathbb{R}}$ has the natural structure of a 2-dimensional real algebraic group (with the group structure inherited from $E$).

**Theorem 2** If $E_{\mathbb{R}}$ is biregularly isomorphic to the product $C_1 \times C_2$ of two real algebraic curves, then necessarily $C_1$ and $C_2$ are projective nonsingular cubics in $\mathbb{P}^2(\mathbb{R})$ (and hence are 1-dimensional real algebraic groups). A biregular isomorphism $C_1 \times C_2 \to E_{\mathbb{R}}$ can be chosen, which is a group isomorphism.

The next task is to identify all (unordered) pairs of cubics $C_1, C_2$, such that the product $C_1 \times C_2$ is biregularly isomorphic to a given $E_{\mathbb{R}}$. First we have to recall a few facts about real cubic curves in $\mathbb{P}^2(\mathbb{R})$.

Given $\alpha \in \mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$, let $\tau_\alpha = \frac{1}{2}(1 + \alpha \sqrt{-1})$ and define

$$
D_\alpha = \{[x : y : z] \in \mathbb{P}^2(\mathbb{R}) \mid zy^2 = 4x^3 - g_2(\tau_\alpha)xz^2 - g_3(\tau_\alpha)z^3 \},
$$
where, as usual, the $g_i(\tau_\alpha)$ are the numbers (in this case real) defined by
\[
g_2(\tau_\alpha) = 60 \sum_{\omega \in \Lambda'_{\alpha}} \omega^{-1}, \quad g_3(\tau_\alpha) = 140 \sum_{\omega \in \Lambda'_{\alpha}} \omega^{-6},
\]
where $\Lambda_\alpha = \mathbb{Z} + \mathbb{Z}_\tau_\alpha$ is a lattice in $\mathbb{C}$, $\Lambda'_\alpha = \Lambda_\alpha \setminus \{0\}$. Each $D_\alpha$ is a nonsingular connected real cubic curve in $\mathbb{P}^2(\mathbb{R})$. Moreover, $D_\alpha$ and $D_\beta$ are not biregularly isomorphic, whenever $\alpha \neq \beta$, and every nonsingular connected real cubic curve in $\mathbb{P}^2(\mathbb{R})$ is isomorphic to some $D_\alpha$. Observe that the complexification $D_\mathbb{C} \subseteq \mathbb{P}^2(\mathbb{C})$ of every nonsingular real cubic $D \subseteq \mathbb{P}^2(\mathbb{R})$ (that is, the complex cubic defined by the same equation as $D$) is itself nonsingular. Denote by $E_\alpha$ the complexification of $D_\alpha$. For each $\alpha \in \mathbb{R}^+$, $\alpha \neq 1$, there exists precisely one $\beta \in \mathbb{R}^+$, $\beta \neq \alpha$, such that $E_\alpha$ and $E_\beta$ are isomorphic complex curves (one takes $\beta = 1/\alpha$). The corresponding real cubics $D_\alpha$ and $D_\beta$ are then called associated real cubics. They play a special role in our investigations.

**Theorem 3** Let $E$ be a complex elliptic curve with complex multiplication. Assume that the discriminant $\delta(E)$ of $\text{End}(E)$ is odd. Then, for a pair of real algebraic curves $C_1, C_2$, the following conditions are equivalent:

(i) The product $C_1 \times C_2$ is biregularly isomorphic to $E_\mathbb{R}$.

(ii) $\{C_1, C_2\}$ is, up to biregular isomorphism, a pair of associated real cubics $\{D_\alpha, D_{1/\alpha}\}$, for some $\alpha = \sqrt{m/n}$, where $m, n$ are coprime positive integers with $mn = -\delta(E)$.

The next statement follows immediately from Theorem 3.

**Corollary 4** Let $E$ be a complex elliptic curve with complex multiplication. Assume that $-\delta(E)$ is the product of powers of $k + 1$ distinct odd primes. Then, up to isomorphism, there are precisely $2^k$ unordered pairs $\{C_1, C_2\}$ of real algebraic curves, such that $E_\mathbb{R}$ is biregularly isomorphic to $C_1 \times C_2$.

**Example 5.** There exist, up to isomorphism, exactly 8 complex elliptic curves defined over $\mathbb{Q}$, such that their underlying real algebraic structure is biregularly isomorphic to the product of two real algebraic curves. Indeed, let

$$
\Omega = \{3, 7, 11, 19, 27, 43, 67, 163\}
$$
and consider $E_{\sqrt{\tau}}$, for $k$ in $\Omega$. It is known (cf. [4] p.233 or [5] p.192) that the curves $E_{\sqrt{\tau}}$ are the only complex elliptic curves defined over $\mathbb{Q}$ with complex
multiplication, such that the discriminants of their rings of endomorphisms are odd (in fact, one has \(\delta(E_{\sqrt{\tau}}) = -k\)). It follows from Theorem 3 that \((E_{\sqrt{\tau}})_\mathbb{R}\) is isomorphic to \(D_{\sqrt{\tau}} \times D_{1/\sqrt{\tau}}\) (even as real algebraic groups), and that this presentation as a product is unique. \(\square\) \(\square\)

Theorems 1 and 2 are proved in Section 3. Theorem 3 is proved in Section 4.

2 A convenient complexification of the underlying real algebraic structure

In this section we recall the construction of an intrinsic complexification of the underlying real algebraic structure of a complex algebraic variety, as described in [3] (essentially due to A. Weil).

A projective complex algebraic variety \(W\) together with an antiholomorphic involution \(\sigma\) will be said to be defined over \(\mathbb{R}\). It is well known that if \((W, \sigma)\) is defined over \(\mathbb{R}\), then there exist a complex algebraic subvariety \(X\) of \(\mathbb{P}^k(\mathbb{C})\), for some \(k\), and a complex isomorphism \(f: W \to X\) such that \(X\) is defined by polynomials with real coefficients, that is, \(\sigma_k(X) = X\), where \(\sigma_k\) is the involution on \(\mathbb{P}^k(\mathbb{C})\) given by complex conjugation, and \(\sigma_k \circ f = f \circ \sigma\). Clearly, \(f\) maps the set \(W_\sigma(\mathbb{R})\) of fixed points of \(\sigma\), called the real part of \((W, \sigma)\), onto \(X \cap \mathbb{P}^k(\mathbb{R})\). Thus \(W_\sigma(\mathbb{R})\) can be considered in the natural way as a real algebraic variety. In fact, \(W_\sigma(\mathbb{R})\), or \(W(\mathbb{R})\), when it is clear which involution \(\sigma\) is meant, is a real algebraic subvariety of \(W_\mathbb{R}\). If \(W_\sigma(\mathbb{R})\) is Zariski dense in \(W\), then we say that \(W\) is a complexification of \(W_\sigma(\mathbb{R})\).

Given a projective complex algebraic variety \(V \subseteq \mathbb{P}^n(\mathbb{C})\), set \(\overline{V} = \sigma_n(V)\). Observe that the mapping \(\gamma_V: V \times \overline{V} \to V \times \overline{V}\), defined by \(\gamma_V(x, y) = (\sigma_n(y), \sigma_n(x))\), is an antiholomorphic involution of \(V \times \overline{V}\). Thus \((V \times \overline{V}, \gamma_V)\) is a projective complex algebraic variety defined over \(\mathbb{R}\). Observe that its real part \((V \times \overline{V})(\mathbb{R})\) is biregularly isomorphic to \(V_\mathbb{R}\). Indeed, the mapping \(h_V: V_\mathbb{R} \to (V \times \overline{V})(\mathbb{R})\),
defined by $h_V(x) = (x, \sigma_n(x))$, is a biregular isomorphism of real algebraic varieties. Since $(V \times \overline{V})(\mathbb{R})$ is Zariski dense in $V \times \overline{V}$, it follows that, identifying $V_{\mathbb{R}}$ and $(V \times \overline{V})(\mathbb{R})$ through $h_V$, one can consider $V \times \overline{V}$ as a complexification of $V_{\mathbb{R}}$.

3 Proofs of Theorems 1 and 2

Lemma 6 If $\Lambda$ and $\Lambda'$ are lattices in $\mathbb{R}^n$ and $L: \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping with $L(\Lambda) \subseteq \Lambda'$, then the induced mapping of the $n$-tori $\tilde{L}: \mathbb{R}^n/\Lambda \to \mathbb{R}^n/\Lambda'$ has topological degree equal to

$$
\frac{\det L |\Lambda|}{|\Lambda'|}.
$$

where the orientation on $\mathbb{R}^n/\Lambda$ and $\mathbb{R}^n/\Lambda'$ is inherited from $\mathbb{R}^n$, and $|\Lambda|$ is the volume of a fundamental parallelogram of $\Lambda$.

Proof. Easy exercise. \hfill \Box

Lemma 7 Let $\alpha = \sqrt{m/n}$, with $m, n$ coprime odd positive integers such that $m - n \equiv 2 \pmod{4}$.

Then there is a biregular isomorphism $f: (E_{\alpha})_{\mathbb{R}} \to D_{\alpha} \times D_{1/\alpha}$ from the underlying real algebraic structure $(E_{\alpha})_{\mathbb{R}}$ of $E_{\alpha}$ onto the product $D_{\alpha} \times D_{1/\alpha}$ of the associated real cubics $D_{\alpha}, D_{1/\alpha}$, which is also a group isomorphism.

Proof. As usual, we identify the quotient torus $\mathbb{C}/\Lambda_{\alpha}$ with the elliptic curve $E_{\alpha} \subseteq \mathbb{P}^2(\mathbb{C})$. If $\pi_{\alpha}: \mathbb{C} \to \mathbb{C}/\Lambda_{\alpha}$ is the canonical projection, then, under the above identification,

$$
\pi_{\alpha}(\mathbb{R}) = D_{\alpha}.
$$

Let $\beta = \frac{1}{2}(m + \alpha \sqrt{-1})$. The assumption about $\alpha$ implies that $\beta \Lambda_{1/\alpha} \subseteq \Lambda_{\alpha}$. Hence $\beta \Lambda_{1/\alpha} \subseteq \overline{\Lambda_{\alpha}}$. The $\mathbb{C}$-linear mapping $\psi: \mathbb{C}^2 \to \mathbb{C}^2$, defined by $\psi(x, y) = (x + \beta y, x + \overline{\beta y})$, maps the lattice $\Lambda_{\alpha} \times \Lambda_{1/\alpha}$ into $\Lambda_{\alpha} \times \overline{\Lambda_{\alpha}}$. Hence it induces a complex morphism

$$
\tilde{\psi}: E_{\alpha} \times E_{1/\alpha} \longrightarrow E_{\alpha} \times \overline{E_{\alpha}},
$$

5
with $\overline{\psi}(D_\alpha \times D_{1/\alpha}) \subseteq (E_\alpha \times \overline{E_\alpha})(\mathbb{R})$. Therefore, the restriction

$$\varphi: D_\alpha \times D_{1/\alpha} \longrightarrow (E_\alpha \times \overline{E_\alpha})(\mathbb{R})$$

of $\overline{\psi}$ to $D_\alpha \times D_{1/\alpha}$ is a regular morphism of real algebraic varieties. We claim that $\varphi$ is an isomorphism. To show the claim it suffices to prove that $\overline{\psi}$ is an isomorphism of complex abelian surfaces, or equivalently, that $\deg \overline{\psi} = 1$. Applying Lemma 6 one has

$$\deg \overline{\psi} = \left| \det \left( \begin{array}{cc} 1 & \beta \\ 1 & \overline{\beta} \end{array} \right) \right|^2 \frac{|\Lambda_\alpha| \cdot |\Lambda_{1/\alpha}|}{|\Lambda_\alpha|^2} = 1.$$  

Since $(E_\alpha)_\mathbb{R}$ is biregularly isomorphic to $(E_\alpha \times \overline{E_\alpha})(\mathbb{R})$ (cf. Section 2), the proof of Lemma 7 is complete.

We shall also need the following result, proved in [3].

**Theorem 8** Let $E$ and $F$ be complex elliptic curves. Assume that $E$ has complex multiplication. Then the following conditions are equivalent:

(i) $E_\mathbb{R}$ and $F_\mathbb{R}$ are biregularly isomorphic (as real algebraic surfaces).

(ii) $F$ has complex multiplication and $\delta(E) = \delta(F)$.  

**Proof of Theorem 1.** The implication (i) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (ii) is proved in [2].

(ii) $\Rightarrow$ (i). Assume that $E$ has complex multiplication and $\delta(E)$ is odd. Set $\alpha = \sqrt{-\delta(E)}$. Since $\delta(E) = \delta(E_\alpha)$, it follows from Theorem 8 that $E_\mathbb{R}$ and $(E_\alpha)_\mathbb{R}$ are biregularly isomorphic. By Lemma 7, $(E_\alpha)_\mathbb{R}$ is biregularly isomorphic to $D_\alpha \times D_{1/\alpha}$. This completes the proof of Theorem 1.  

**Proof of Theorem 2.** Assume that $E_\mathbb{R}$ is biregularly isomorphic to the product $C_1 \times C_2$ of two real algebraic curves $C_1, C_2$. Without loss of generality we may assume that $C_j$, $j = 1, 2$, is the real part of a nonsingular complex projective curve $C_j \mathbb{C}$. Since $C_1 \times C_2$, $E_\mathbb{R}$ and $(E \times \overline{E})(\mathbb{R})$ are biregularly isomorphic real algebraic surfaces (cf. Section 2), it follows that the complex varieties $C_1 \mathbb{C} \times C_2 \mathbb{C}$ and $E \times \overline{E}$ are birationally isomorphic. This immediately implies that the curves $C_1 \mathbb{C}$ and $C_2 \mathbb{C}$ have genus 1, which concludes the first part of Theorem 2. The second part of the theorem follows from well known properties of rational mappings between complex abelian varieties.  

6
4 Proof of Theorem 3

Given a system \( \{a_1, a_2, a_3, a_4\} \) of 4 vectors in \( \mathbb{C}^2 \), linearly independent over \( \mathbb{R} \), let us denote the lattice \( \mathbb{Z}a_1 + \mathbb{Z}a_2 + \mathbb{Z}a_3 + \mathbb{Z}a_4 \) by

\[
\mathbb{Z}\langle a_1, \ldots, a_4 \rangle.
\]

Proof of Theorem 3. (ii) \( \Rightarrow \) (i). Let \( \alpha \) be as in condition (ii). Then, by Lemma 7, \( (E_\alpha)_\mathbb{R} \) is biregularly isomorphic to \( D_\alpha \times D_1/\alpha \). Since \( \delta(E_\alpha) = -mn = \delta(E) \), it follows from Theorem 8 that \( (E_\alpha)_\mathbb{R} \) and \( E_\mathbb{R} \) are isomorphic too. This implies (i).

(i) \( \Rightarrow \) (ii). Let \( E \) be a complex elliptic curve such that \( E_\mathbb{R} \) is biregularly isomorphic to \( C_1 \times C_2 \), and let \( \beta = \sqrt{-\delta(E)} \). Theorem 2 implies that \( C_1 \) and \( C_2 \) are real cubic curves in \( \mathbb{P}^2(\mathbb{R}) \). Therefore, there exist positive real numbers \( \alpha_1 \) and \( \alpha_2 \), such that \( C_j = D_{\alpha_j} \), \( j = 1, 2 \). Since \( E \) has complex multiplication, \( \alpha_1^2 \) and \( \alpha_2^2 \) are necessarily in \( \mathbb{Q} \). As real varieties, \( E_\mathbb{R} \) and \( (E_\beta)_\mathbb{R} \) are biregularly isomorphic (because \( \delta(E) = \delta(E_\beta) \), cf. Theorem 8).

Hence we may assume that \( E = E_\beta \). In particular, \( E = \overline{E} \).

Set

\[
\Lambda = \mathbb{Z} \left\langle \left( \begin{array}{c} 1 \\ 0 \\ \frac{1}{2}(1 + \alpha_1 i) \\ 0 \\ \frac{1}{2}(1 + \alpha_2 i) \end{array} \right) \right\rangle,
\]

where \( i = \sqrt{-1} \). Clearly, \( \overline{\Lambda} = \Lambda \). Observe now that \( C_1 \times C_2 \) is biregularly isomorphic to the real part \( (\mathbb{C}^2/\Lambda)(\mathbb{R}) \) of \( \mathbb{C}^2/\Lambda \), with respect to the antiholomorphic involution on \( \mathbb{C}^2/\Lambda \) induced by the standard conjugation on \( \mathbb{C}^2 \).

Set

\[
\Lambda' = \mathbb{Z} \left\langle \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2i} \\ 0 \\ \frac{1}{2i} \\ \frac{i}{2} \\ \frac{i}{2} \end{array} \right) \right\rangle,
\]

where \( \tau = \frac{1}{2}(1 + \beta i) \). Note that \( \Lambda' = L_1(\Lambda_\beta \times \overline{\Lambda}_\beta) \), where \( L_1: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) is defined by \( L_1(z, w) = (\frac{1}{2}(z + w), \frac{1}{2i}(z - w)) \). Again \( \overline{\Lambda} = \Lambda' \).

Consider, \( E \times \overline{E} = E \times E \) endowed with the involution \( \gamma_E \) defined in Section 2. Let \( \mathbb{C}^2/\Lambda' \) be endowed with the involution induced by the standard conjugation on \( \mathbb{C}^2 \). Then \( L_1 \) induces an equivariant isomorphism of abelian varieties

\[
E \times \overline{E} \longrightarrow \mathbb{C}^2/\Lambda'.
\]
Since \( C_1 \times C_2, E_\mathbb{R}, \) and \( (E \times E)(\mathbb{R}) \) are isomorphic as real algebraic groups (cf. Theorem 2 and Section 2), it follows from the above constructions that there is an isomorphism

\[
\mathbb{C}^2 / \Lambda \rightarrow \mathbb{C}^2 / \Lambda',
\]

induced by a \( \mathbb{C} \)-linear automorphism \( L_2 \) of \( \mathbb{C}^2 \), defined over \( \mathbb{R} \). In particular, \( L_2(\Lambda) = \Lambda' \).

Hence, for the lattices in \( \mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2 \)

\[
\widehat{\Lambda} = \Lambda \cap \mathbb{R}^2 + \Lambda \cap i\mathbb{R}^2 = \mathbb{Z} \left\langle \begin{pmatrix} 1 \\ 0 \\ \alpha_1 \\ 0 \\ \alpha_2 \end{pmatrix} \right\rangle,
\]

and

\[
\widehat{\Lambda'} = \Lambda' \cap \mathbb{R}^2 + \Lambda' \cap i\mathbb{R}^2 = \mathbb{Z} \left\langle \begin{pmatrix} 1 \\ 0 \\ \text{Re } \tau \\ 0 \\ \text{Im } \tau \\ -\text{Re } \tau \end{pmatrix} \right\rangle,
\]

one has \( L_2(\widehat{\Lambda}) = \widehat{\Lambda'} \). Denote

\[
L = \begin{pmatrix} 1 & \text{Re } \tau \\ 0 & \text{Im } \tau \end{pmatrix}^{-1}
\]

and set \( A = L \circ L_2 \). Then

\[
A(\widehat{\Lambda}) = L(\widehat{\Lambda'}) = \mathbb{Z} \left\langle \begin{pmatrix} 1 \\ 0 \\ \text{Re } \tau \\ 0 \\ \text{Im } \tau \end{pmatrix} \right\rangle,
\]

Since

\[
A(\mathbb{Z}^2) = A(\widehat{\Lambda} \cap \mathbb{R}^2) = L(\widehat{\Lambda'}) \cap \mathbb{R}^2 = \mathbb{Z}^2,
\]

one has necessarily \( A \in \text{GL}_2(\mathbb{Z}) \), \( \det A = \pm 1 \) and

\[
A(\widehat{\Lambda} \cap i\mathbb{R}^2) = L(\widehat{\Lambda'}) \cap i\mathbb{R}^2 \quad (1)
\]

Denoting the volume of the fundamental parallelogram of a lattice \( \Omega \) by \( |\Omega| \), one has from (1)

\[
\alpha_1 \alpha_2 = |\det A| : |\widehat{\Lambda} \cap i\mathbb{R}^2| = |L(\widehat{\Lambda'}) \cap i\mathbb{R}^2| = \left| \frac{(\text{Re } \tau)^2 - |\tau|^2}{(\text{Im } \tau)^2} \right| = 1.
\]

Hence \( \alpha_1 = 1/\alpha_2 \), that is \( C_1 \) and \( C_2 \) are associated real cubics.
By construction, $\alpha_1 = \sqrt{m/n}$, for some coprime positive integers $m, n$. We claim that $\delta(E) = -mn$. Indeed, since $E_\mathbb{R}$ is biregularly isomorphic to $D_{\alpha_1} \times D_{1/\alpha_1}$, it follows that $E \times E$ is isomorphic to $E_{\alpha_1} \times E_{1/\alpha_1}$, and hence to $E_{\alpha_1} \times E_{\alpha_1}$. This implies that the rings of endomorphisms $\text{End}(E_\mathbb{R} \times E_\mathbb{R})$ and $\text{End}(E_{\alpha_1} \times E_{\alpha_1})$ are isomorphic. In particular, their centers, isomorphic to $\text{End}(E)_{\alpha_1}$ and $\text{End}(E)_{\alpha_1}$, respectively, are isomorphic. Therefore $\delta(E) = \delta(E_{\alpha_1}) = -mn$, which shows the claim.

This completes the proof of Theorem 3. 

\textbf{References}


