On the space of real line arrangements

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Abstract

The set of all real line arrangements of given degree in the real projective plane is known to have a natural semialgebraic structure. The nonreduced arrangements are singular points of this structure. We show that the set of all real line arrangements of given degree has also a natural structure of a smooth compact connected affine real algebraic variety. As a consequence, we get a projectively linear structure on the set of all real line arrangements of given degree.

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1 Introduction

Let $K$ be any field. A line arrangement over $K$ is a closed subscheme [2] of the projective plane $\mathbb{P}^2 = \mathbb{P}^2_K$, defined by a nonzero homogeneous polynomial $F \in K[X, Y, Z]$ that is equal to the product of its linear factors in $K[X, Y, Z]$. Equivalently, a line arrangement over $K$ is a proper closed subscheme of $\mathbb{P}^2$ that is the scheme-theoretic union of finitely many projective lines in $\mathbb{P}^2$. Note that, with the current definition, line arrangements are not necessarily reduced or nonempty (cf. [3]).

Since the set of projective lines in $\mathbb{P}^2$ is parametrized by the set $(\mathbb{P}^2)^\vee(K)$ of all $K$-rational points of the dual projective plane $(\mathbb{P}^2)^\vee$, the set of all line arrangements of degree $d$ over $K$ is naturally parametrized by the symmetric power

$$\mathcal{A}_d = ( (\mathbb{P}^2)^\vee(K) )^{(d)} ,$$

where $d$ is a natural integer.

Now, the set $\mathcal{A}_d$ has two bad properties:

1. $\mathcal{A}_d$ is not, in a natural way, the set of $K$-rational points of an algebraic variety over $K$, and
2. $\mathcal{A}_d$ contains singularities, as a subset of the algebraic variety $((\mathbb{P}^2)^\gamma)^{(d)}$.

Indeed, as for property 1, $\mathcal{A}_d$ is a subset of the set $((\mathbb{P}^2)^\gamma)^{(d)}(K)$ of all $K$-rational points of the symmetric power $((\mathbb{P}^2)^\gamma)^{(d)}$. One has a strict inclusion

$$\mathcal{A}_d \subsetneq ((\mathbb{P}^2)^\gamma)^{(d)}(K)$$

if and only if the field $K$ admits a nontrivial extension of degree $\leq d$. For example, if $K$ is algebraically closed then $\mathcal{A}_d$ is equal to the set of $K$-rational points of $((\mathbb{P}^2)^\gamma)^{(d)}$. However, if $K$ is not algebraically closed, $\mathcal{A}_d$ is strictly contained in $((\mathbb{P}^2)^\gamma)^{(d)}(K)$ for all $d \geq d_0$, for some natural integer $d_0$. For example, if $K$ is the field $\mathbb{R}$ of real numbers then $\mathcal{A}_d$ is a strict semialgebraic subset of $((\mathbb{P}^2)^\gamma)^{(d)}(\mathbb{R})$ for all $d \geq 2$.

As for property 2, since $(\mathbb{P}^2)^\gamma$ is 2-dimensional, the symmetric power $((\mathbb{P}^2)^\gamma)^{(d)}$ is singular along the so-called big diagonal $\Delta$ for all $d \geq 2$.

While seemingly nothing can be done to resolve property 1, one can resolve property 2 by resolution of singularities. This has, however, the following drawback. Let $\mathcal{A}_d$ be a resolution of singularities of $\mathcal{A}_d$. Let $\mathcal{A}$ be the disjoint union of $\mathcal{A}_d$, for $d \in \mathbb{N}$ and let, similarly, $\mathcal{A}$ be the disjoint union of $\mathcal{A}_d$, for $d \in \mathbb{N}$. Then, the scheme-theoretic union of line arrangements over $K$, which is a monoid law

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

on $\mathcal{A}$, does not extend to a map

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}.$$

Therefore, even if $K$ is algebraically closed, property 2 erects serious obstacles.

The object of this paper is to show that, when $K$ is the field $\mathbb{R}$ of real numbers, both bad properties 1 and 2 can be resolved. More precisely, we show that $\mathcal{A}_d$ can be identified, in a natural way, with the whole set of real points of a proper smooth algebraic variety over $\mathbb{R}$ (see Corollary 2.2). In particular, $\mathcal{A}_d$ has a natural structure of a smooth compact connected affine real algebraic variety in the sense of [1]. Moreover, with respect to this structure, the scheme-theoretic union of real line arrangements

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

is an algebraic map (see Corollary 2.3).

The idea is that the set $\mathcal{A}_d$ is parametrized by the set of all effective divisors of degree $2d$ on a certain real algebraic curve $Q$ (see Section 2). In Section 3, we determine more explicitly this parametrization as a parametrization of $\mathcal{A}_d$ by Laurent polynomials.
A real algebraic structure on the space of real line arrangements

Let $Q$ be the anisotropic real conic in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{R}}$ defined by the equation

$$X^2 + Y^2 + Z^2 = 0.$$  

To say that $Q$ is anisotropic means that $Q$ has no real points, i.e., $Q(\mathbb{R}) = \emptyset$. Define, for any natural integer $d$,

$$\text{Div}_{\geq 0}^d(Q)$$

to be the set of effective divisors on $Q$ of degree $2d$. Let

$$\text{Div}_{\geq 0}(Q) = \prod_{d \in \mathbb{N}} \text{Div}_{\geq 0}^d(Q)$$

to be the disjoint union of $\text{Div}_{\geq 0}^d(Q)$, for $d \in \mathbb{N}$. Since all divisors on $Q$ are of even degree, the set $\text{Div}_{\geq 0}(Q)$ is nothing but the set of all effective divisors on $Q$, as is suggested by its notation. The addition of effective divisors defines a monoid law

$$\text{Div}_{\geq 0}(Q) \times \text{Div}_{\geq 0}(Q) \rightarrow \text{Div}_{\geq 0}(Q)$$
on $\text{Div}_{\geq 0}(Q)$.

Define a map

$$\varphi : A \rightarrow \text{Div}_{\geq 0}(Q)$$
as follows. Let $A$ be a real line arrangement in $\mathbb{P}^2$. Then, the intersection product $A \cdot Q$ is well defined since no irreducible component of $A$ is contained in $Q$. Therefore, the intersection product $A \cdot Q$ is a well defined effective divisor on $Q$. Define

$$\varphi(A) = A \cdot Q.$$  

**Theorem 2.1.** The map $\varphi$ is an isomorphism of graded monoids, i.e., $\varphi$ is a bijective morphism of monoids such that

$$\varphi(A_d) = \text{Div}_{\geq 0}^d(Q)$$

for all $d \in \mathbb{N}$.

**Proof.** Let $A$ and $B$ be two real line arrangements. Denote by $A + B$ the scheme-theoretic union of $A$ and $B$. One has

$$\varphi(A + B) = (A + B) \cdot Q = A \cdot Q + B \cdot Q = \varphi(A) + \varphi(B).$$
Therefore, \( \varphi \) is a morphism of monoids. Moreover, \( \varphi \) is a morphism of graded monoids since

\[
\varphi(A_d) \subseteq \text{Div}^{2d}_{\geq 0}(Q),
\]

by Bezout’s Theorem.

In order to show that \( \varphi \) is an isomorphism, it suffices to show that the restriction \( \varphi_d \) of \( \varphi \) to \( A_d \) is bijective onto \( \text{Div}^{2d}_{\geq 0}(Q) \), for any \( d \in \mathbb{N} \). Choose \( D \in \text{Div}^{2d}_{\geq 0}(Q) \). There are distinct closed points \( P_1, \ldots, P_n \) of \( Q \) and nonzero natural integers \( m_1, \ldots, m_n \) such that

\[
D = \sum_{i=1}^{n} m_i P_i.
\]

For each \( i \in \{1, \ldots, n\} \), there is exactly one real projective line \( L_i \) in \( \mathbb{P}^2 \) such that \( L_i \cdot Q = P_i \). Indeed, a closed point \( P_i \) of \( Q \) corresponds to a pair of distinct complex conjugate points \( \{Q, \overline{Q}_i\} \) of the complexification \( Q_{\mathbb{C}} = Q \times_{\mathbb{R}} \mathbb{C} \) of \( Q \). The real projective line \( L_i \) is the unique real projective line whose complexification passes through \( Q_i \) and \( \overline{Q}_i \). Let \( A \) be the real line arrangement \( \sum m_i L_i \). Then \( A \in A_d \) and \( \varphi_d(A) = D \). This shows surjectivity of \( \varphi_d \). Moreover, one easily sees that \( A \) is the unique real line arrangement in \( A_d \) satisfying \( \varphi_d(A) = D \). Hence, \( \varphi_d \) is also injective.

Since \( Q \) is a rational curve over \( \mathbb{R} \), the set \( \text{Div}^{2d}_{\geq 0} \) can be naturally identified with the set of real points of a real projective space. Indeed, let \( \mathcal{L}(d) \) be the restriction to \( Q \) of the invertible sheaf \( \mathcal{O}(d) \) on \( \mathbb{P}^2 \), for any \( d \in \mathbb{N} \). The map

\[
\psi_d : \mathbb{P}(H^0(Q, \mathcal{L}(d))) \longrightarrow \text{Div}^{2d}_{\geq 0}(Q)
\]

that associates to a nonzero global section \( s \) of \( \mathcal{L}(d) \) its divisor \( \text{div}(s) \), is a bijection by the Riemann-Roch Theorem. Here, the notation \( \mathbb{P}(V) \) denotes the real projective space of all 1-dimensional subspaces of the real vector space \( V \).

**Corollary 2.2.** Let \( d \in \mathbb{N} \). The map

\[
\psi_d^{-1} \circ \varphi_d : A_d \longrightarrow \mathbb{P}(H^0(Q, \mathcal{L}(d)))
\]

is a bijection. In particular, the set \( A_d \) of all real line arrangements of degree \( d \) has a natural structure of a real algebraic variety in the sense of [1]. With respect to this structure, \( A_d \) is isomorphic to \( \mathbb{P}^{2d}(\mathbb{R}) \). In particular, \( A_d \) is a smooth compact connected affine real algebraic variety.
Let us make precise what is meant by a natural structure of a real algebraic variety on the set $\mathcal{A}_d$. Let $\{A_t\}_{t \in T}$ be an algebraic family of real line arrangements of degree $d$ over a base $T$. More precisely, $T$ is an affine real algebraic variety in the sense of [1], and the subset
\[ A = \bigcup_{t \in T} A_t \times \{t\} \]
of $\mathbb{P}^2 \times T$ is an algebraic subset, each of whose fibers $A_t$ over $t \in T$ is a real line arrangement of degree $d$. More concretely, $A$ is defined by a homogeneous polynomial
\[ F \in \mathcal{R}(T)[X, Y, Z] \]
of degree $d$ with coefficients in the ring $\mathcal{R}(T)$ of all regular functions on $T$ [1], such that for all $t \in T$, the evaluation of $F$ at $t$ defines a real line arrangement in $\mathbb{P}^2$ of degree $d$. To say that the above structure on $\mathcal{A}_d$ of a real algebraic is natural means that the map
\[ f : T \to \mathcal{A}_d \]
defined by $f(t) = A_t$ is a real algebraic morphism.

Note that the situation is rather subtle; the universal family
\[ U_d = \bigcup_{A \in \mathcal{A}_d} A \times \{A\} \subseteq \mathbb{P}^2 \times \mathcal{A}_d \]
of real line arrangements of degree $d$ is not an algebraic family of real line arrangements over $\mathcal{A}_d$. In fact, the subset $U_d$ is only semialgebraic. More precisely, $U$ is defined by a homogeneous polynomial $F$ with coefficients in the ring $\mathcal{S}(\mathcal{A}_d)$ of all semialgebraic functions on $\mathcal{A}_d$, and not with coefficients in $\mathcal{R}(\mathcal{A}_d)$. This will be proven in the next section (see Proposition 3.1).

Another observation we would like to make is that, by Corollary 2.2, $\mathcal{A}_d$ is isomorphic to $\mathbb{P}^{2d}(\mathbb{R})$, with respect to its natural real algebraic structure. In particular, one gets a projectively linear structure on the set $\mathcal{A}_d$. For example, given two distinct real line arrangements $A$ and $B$ of degree $d$, there is a unique real projective line of real line arrangements of degree $d$ that contains $A$ and $B$!

We conclude this section by a further consequence of Theorem 2.1. Put
\[ \mathbb{P}(H^0(Q, \mathcal{L}(\ast))) = \prod_{d \in \mathbb{N}} \mathbb{P}(H^0(Q, \mathcal{L}(d))). \]
The tensor product of global sections endows $\mathbb{P}(H^0(Q, \mathcal{L}(\ast)))$ with the structure of a graded monoid. Let
\[ \psi : \mathbb{P}(H^0(Q, \mathcal{L}(\ast))) \rightarrow \text{Div}_{\geq 0}(Q) \]
be the map whose restriction to $\mathbb{P}(H^0(Q, \mathcal{L}(d)))$ is equal to $\psi_d$. 

5
Corollary 2.3. The map
\[ \psi^{-1} \circ \varphi : \mathcal{A} \longrightarrow \mathbb{P}(H^0(Q, \mathcal{L}(\ast))) \]
is an isomorphism of graded monoids. In particular, \( \mathcal{A} \) is a real algebraic monoid, i.e., the scheme-theoretic union on the set of all real line arrangements \( \mathcal{A} \) is real algebraic with respect to the natural real algebraic structure on \( \mathcal{A} \).

3 An explicit description of the real algebraic structure on \( \mathcal{A}_d \)

As observed in Section 2, the real algebraic curve \( Q \) is rational. Hence, its complexification \( Q_\mathbb{C} \) is isomorphic to the complex projective line \( \mathbb{P}^1_\mathbb{C} \). Choose, once and for all, an isomorphism between \( Q_\mathbb{C} \) and \( \mathbb{P}^1_\mathbb{C} \) having the following property. The action of complex conjugation on \( Q_\mathbb{C} \) corresponds to the action of complex conjugation on \( \mathbb{P}^1_\mathbb{C} \) defined by
\[ z \mapsto -\frac{1}{\bar{z}} \]
for \( z \in \mathbb{C} \), where \( z \mapsto \bar{z} \) denotes the usual action of complex conjugation on \( \mathbb{C} \). Such an isomorphism exists since the action of complex conjugation on \( \mathbb{P}^1_\mathbb{C} \) defined above does not have any fixed points on \( \mathbb{P}^1(\mathbb{C}) \). With the point of view of \( \mathbb{P}^1_\mathbb{C} \) as the Riemann sphere \( S^2 = \mathbb{P}^1(\mathbb{C}) \), the action of complex conjugation on \( Q_\mathbb{C} \) corresponds to the antipodal action on \( S^2 \).

Let \( d \in \mathbb{N} \). With respect to the isomorphism \( Q_\mathbb{C} \cong \mathbb{P}^1_\mathbb{C} \), the complexification \( \mathcal{L}(d)_\mathbb{C} \) of the invertible sheaf \( \mathcal{L}(d) \) on \( Q \) is isomorphic to the invertible sheaf \( \mathcal{O}(d \cdot 0 + d \cdot \infty) \) on \( \mathbb{P}^1_\mathbb{C} \). The complex vector space of global sections of \( \mathcal{O}(d \cdot 0 + d \cdot \infty) \) is the complex vector space \( L_\mathbb{C}(d) \) of all complex Laurent polynomials
\[ \sum_{i=-d}^{d} a_i Z^i, \]
where \( a_i \in \mathbb{C} \) for \( i = -d, \ldots, d \). The action of complex conjugation on the set of all global sections of \( \mathcal{L}(d)_\mathbb{C} \) corresponds to the action of complex conjugation on \( L_\mathbb{C}(d) \) defined by
\[ \sum_{i=-d}^{d} a_i Z^i \mapsto \sum_{i=-d}^{d} (-1)^{|i|} \bar{a}_{-i} Z^i, \]
where \( a_i \in \mathbb{C} \) for \( i = -d, \ldots, d \). Therefore, one can identify the real vector space of global sections of \( \mathcal{L}(d) \) with the real vector space \( L(d) \) of all complex
Laurent polynomials
\[ \sum_{i=-d}^{d} a_i Z^i, \]
where the \( a_i \in \mathbb{C} \) satisfy \( a_{-i} = (-1)^i a_i \) for all \( i = 0, \ldots, d \). In particular, we can identify \( \mathbb{P}(H^0(Q, \mathcal{L}(d))) \) with the real projective space \( \mathbb{P}(L_d) \).

The set of effective divisors \( \text{Div}^{2d}_{\geq 0}(Q) \) of degree \( 2d \) on \( Q \) can be identified with the set \( D^{2d} \) of effective divisors of degree \( 2d \) on \( \mathbb{P}^1 \), that are stable for the action of complex conjugation on \( \mathbb{P}^1 \) as defined above.

The map \( \psi_d : \mathbb{P}(H^0(Q, \mathcal{L}(d))) \to \text{Div}^{2d}_{\geq 0}(Q) \) then corresponds to the map
\[ \chi_d : \mathbb{P}(L(d)) \to D^{2d} \]
defined by letting \( \chi(P) \) be the divisor \( \text{div}(P) + d \cdot 0 + d \cdot \infty \), for any Laurent polynomial \( P \in L(d) \). Here, \( \text{div}(P) \) is the divisor of \( P \) as a rational function on \( \mathbb{P}^1 \).

Next, we want to have a more concrete description of the map
\[ \varphi_d^{-1} : \text{Div}^{2d}_{\geq 0}(Q) \to \mathcal{A}_d. \]

The set \( \text{Div}^{2d}_{\geq 0}(Q) \) has already been identified with \( D^{2d} \). We define a map
\[ \rho_d : D^{2d} \to \mathcal{A}_d \]
as follows. An element \( D \) of \( D^{2d} \) is a divisor on the Riemann sphere \( S^2 \) of the form
\[ \sum_{i=1}^{n} m_i (P_i + [-1]P_i), \]
where \([-1]\) is the antipodal map on \( S^2 \), the points \( P_i, [-1]P_i \), for \( i = 1, \ldots, n \), are distinct, and the \( m_i \) are nonzero natural integers. For each \( i \in \{1, \ldots, n\} \)
let \( C_i \subseteq S^2 \) be the great circle of points that are equidistant to \( P_i \) and \([-1]P_i\). Let \( \pi : S^2 \to \mathbb{P}^2(\mathbb{R}) \) be the quotient map for the antipodal action on \( S^2 \). Let \( L_i \) be the real projective line in \( \mathbb{P}^2 \) such that \( L_i(\mathbb{R}) = \pi(C_i) \). Define
\[ \rho_d(D) = \sum_{i=1}^{n} m_i L_i. \]

Then it is an easy matter to check that \( \rho \) corresponds to the map \( \varphi_d^{-1} \), after a suitable change of coordinates on \( \mathbb{P}^2 \).

Resuming, the map
\[ \rho_d \circ \chi_d : \mathbb{P}(L(d)) \to \mathcal{A}_d \]
corresponds under the above identifications to the map $\varphi_d^{-1} \circ \psi_d$. In particular, $\rho_d \circ \chi_d$ is an isomorphism of real algebraic varieties with respect to the natural structure of a real algebraic variety on $A_d$.

As an application we show the following statement.

**Proposition 3.1.** Let $d \in \mathbb{N}$. The universal family $U_d$ of real line arrangements of degree $d$ is semi-algebraic. If $d \geq 2$ then $U_d$ is not algebraic.

**Proof.** The universal family of effective divisors of degree $2d$ on $Q$ is clearly algebraic. Therefore, the universal family of divisors in $D^{2d}$ on the Riemann sphere $S^2$ is algebraic as well. Then the universal family of arrangements of great circles in $S^2$ of degree $d$ is semi-algebraic. Hence, the universal family $U_d$ of real line arrangements of degree $d$ is semi-algebraic.

We show that $U_d$ is truely semi-algebraic, i.e., nonalgebraic if $d \geq 2$. By Corollary 2.3, it suffices to show this for $d = 2$.

Define a family of complex polynomials $P_t \in \mathbb{C}[Z]$, depending on a real parameter $t \in \mathbb{R}$, by

$$P_t = Z^2 + t.$$ Symmetrize $P_t$ multiplicatively to a family of Laurent polynomials in $L(2)$:

$$L_t = tZ^{-2} + (t^2 + 1) + tZ^2.$$ for $t \in \mathbb{R}$.

Let $C_t$ be the associated family of arrangements of great circles of degree 2 in $S^2$, and let $A_t$ be the associated family of real line arrangements of degree 2.

For $t < 0$, the roots of $P_t$ are on the real axis. Therefore, the divisor $\text{div}(L_t) + 2 \cdot 0 + 2 \cdot \infty$ has its support on the real axis as well, for $t < 0$. It follows that the points $\pm \sqrt{-1}$ belong to $C_t$ for $t < 0$. In fact, the points $\pm \sqrt{-1}$ are the intersection points of the two great circles of $C_t$, for $t < 0$. Hence, the point $\pi(\pm \sqrt{-1}) \in \mathbb{P}^2(\mathbb{R})$ belongs to $A_t$ when $t < 0$.

Now suppose that $U_2$ is algebraic. Then $A_t$ is algebraic too, and hence the point $\pi(\pm \sqrt{-1})$ belongs to $A_t$ for $t \geq 0$ as well. But for $t = 1$, the roots of $P_t$ are $\pm \sqrt{-1}$. Therefore,

$$\text{div}(L_1) + 2 \cdot 0 + 2 \cdot \infty = 2 \cdot \sqrt{-1} + 2 \cdot (-\sqrt{-1}).$$

Then, $C_1$ is equal to $2 \cdot \mathbb{P}^1(\mathbb{R})$, where $\mathbb{P}^1(\mathbb{R})$ is considered as a subset of $\mathbb{P}^1(\mathbb{C}) = S^2$. It follows that $C_1$ does not contain any of the points $\pm \sqrt{-1}$. Then, $A_1$ does not contain the point $\pi(\pm \sqrt{-1})$ of $\mathbb{P}^2(\mathbb{R})$. Contradiction. \qed
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