Heights on abelian varieties

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1 Height on projective space

We will follow to a great extend Chapter 6 of [5].

If $K$ is a number field and $v$ a finite place of $K$, that is, $v$ corresponds to a prime ideal $p$ of the ring of integers of $K$, then we define a norm $\| \cdot \|_v$ on $K$ by

$$\|x\|_v = \left( \frac{1}{N_p} \right)^{\text{ord}_v(x)},$$

where $N_p$ is the absolute norm of $p$. If $v$ is an infinite place, that is, $v$ corresponds to an embedding $\sigma$ of $K$ in $\mathbb{R}$ or $v$ corresponds to a conjugate pair $\{\sigma, \overline{\sigma}\}$ of embeddings of $K$ in $\mathbb{C}$, then we define a norm $\| \cdot \|_v$ on $K$ by

$$\|x\|_v = \begin{cases} \left| \sigma(x) \right|, & \text{if } v \text{ is real,} \\ \left| \sigma(x) \right|^2, & \text{if } v \text{ is complex.} \end{cases}$$

Clearly, for any place $v$ of $K$, the homothety $y \mapsto xy$ transforms a Haar measure $\mu$ on the completion $K_v$ of $K$ at $v$ into $\|x\|_v \cdot \mu$. Let $M_K$ be the set of places of $K$ and $M_K^\infty$ the set of infinite places. Then we have the product formula

$$\prod_{v \in M_K} \|x\|_v = 1,$$

for every $x \in K^*$. This can be easily seen as follows (cf. A. Weil: Basic Number Theory, Ch. IV, Sect. 4, Theorem 5). Let $A$ be the ring of adeles of $K$. The product formula will follow if we prove that any Haar measure on
A is invariant under the homothety $\lambda_x : y \mapsto xy$ of $A$, for any $x \in K^*$. Since $A/K$ is a compact topological group and $\lambda_x$ induces an isomorphism $\overline{\lambda_x}$ of $A/K$, any Haar measure on $A/K$ is invariant under $\overline{\lambda_x}$. Moreover, since $K$ is discrete and the restriction $\overline{\lambda_x}$ of $\lambda_x$ to $K$ is an isomorphism of $K$ as a topological group, any Haar measure on $K$ is invariant under $\overline{\lambda_x}$. Therefore, any Haar measure on $A$ is invariant under $\lambda_x$.

If $P = (x_0 : \cdots : x_n)$ is in $\mathbb{P}^n(K)$ then we define the height of $P$ relative to $K$ by

$$H_K(P) = \prod_{v \in M_K} \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}.$$ 

Observe that, by the product formula, this is well defined.

**Example 1.** If $K = \mathbb{Q}$ and $P \in \mathbb{P}^n(\mathbb{Q})$ then we may assume $P = (x_0 : \cdots : x_n)$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$. Then

$$H_{\mathbb{Q}}(P) = \max\{|x_0|, \ldots, |x_n|\}.$$ 

$\square$

It is clear that $H_K(P) \geq 1$, for every $P \in \mathbb{P}^n(K)$. If $L$ is a finite extension of $K$ and $P \in \mathbb{P}^n(K)$ then

$$H_L(P) = H_K(P)^{[L:K]}.$$

Hence, we can define the absolute height $H$ on $\mathbb{P}^n(\overline{K})$ by

$$H(P) = H_L(P)^{1/[L:K]},$$

where $L$ is some number field containing the coordinates of $P$. It will be convenient to define the (logarithmic) height $h$ on $\mathbb{P}^n(\overline{K})$ by

$$h(P) = \log H(P).$$

**Theorem 2 (Northcott)** Let $C$ and $d$ be constants. Then

$$\{P \in \mathbb{P}^n(\overline{K}) \mid H(P) \leq C[K(P) : K] \leq d\}$$

is a finite set.

For a proof the reader is referred to [5], Chapter 6 or [32].
2 Heights on projective varieties

We will define height functions on a projective algebraic variety $V$ over a number field $K$, using morphisms from $V$ into projective space. Suppose

$$f : V \rightarrow \mathbb{P}^n$$

is a morphism of algebraic varieties over $K$. Then one defines the (logarithmic) height on $V$ relative to $f$ by

$$h_f : V(\overline{K}) \rightarrow \mathbb{R}$$

$$P \mapsto h(f(P)).$$

Let us call real-valued functions $h$ and $h'$ on the set $V(\overline{K})$ equivalent, denoted by $h \sim h'$, if $|h-h'|$ is bounded on $V(\overline{K})$. It turns out that the height $h_f$ depends only, up to equivalence, on the invertible sheaf $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$.

**Theorem 3** Let $V$ be a projective algebraic variety over $K$. If $f : V \rightarrow \mathbb{P}^n$ and $g : V \rightarrow \mathbb{P}^m$ are morphisms over $K$ such that

$$f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong g^*\mathcal{O}_{\mathbb{P}^m}(1)$$

then $h_f$ and $h_g$ are equivalent.

**Proof.** Recall that a morphism $\varphi : V \rightarrow \mathbb{P}^k$ is uniquely determined by (the isomorphism class of) the invertible sheaf $\mathcal{L} = \varphi^*(1)$ and the global sections $s_i = \varphi^*x_i \in \Gamma(V, \mathcal{L})$. Therefore, it suffices to prove the theorem in the case that $m \geq n$ and $f = \pi \circ g$, where $\pi$ is the rational map

$$\pi : \mathbb{P}^m \rightarrow \mathbb{P}^n$$

$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_n).$$

Clearly, $h_g - h_f \geq 0$. To prove that $h_g - h_f$ is bounded from above, observe that

$$g(V) \cap \mathfrak{V}(x_0, \ldots, x_n)$$

is empty. Since $g(V)$ is closed, let $I \subseteq K[x_0, \ldots, x_m]$ be its defining homogeneous ideal. Then

$$\sqrt{I + (X_0, \ldots, X_n)} = (X_0, \ldots, X_n)$$

is the ideal of $\mathfrak{V}(x_0, \ldots, x_n)$. Hence, $h_g - h_f$ is bounded by the height of $I$. Since $I$ is homogeneous, $h_g - h_f$ is bounded by the height of $I$. Therefore, $h_g - h_f$ is bounded.
in \( K[X_0, \ldots, X_m] \). Therefore, there exist a positive integer \( q \) and \( F_{ij} \in K[X_0, \ldots, X_m] \) such that

\[
X_{n+i}^q - \sum_{j=0}^{n} F_{ij} X_j \in I, \quad i = 0, \ldots, m - n.
\]

We may assume \( F_{ij} \) to be homogeneous of degree \( q - 1 \). Denote the coefficients of \( F_{ij} \) by \( a_{ijk} \). If \( L \subseteq K \) is a finite extension of \( K \) and \( w \) is a place of \( L \) then we define

\[
\varepsilon_w = \begin{cases} 
0 & \text{if } w \text{ is finite}, \\
1 & \text{if } w \text{ is real}, \\
2 & \text{if } w \text{ is complex}
\end{cases}
\]

and put

\[
c_w = (n + 1)^{\varepsilon_w} \left( \frac{q - 1 + m}{m} \right)^{\varepsilon_w} \cdot \max \| a_{ijk} \|_w.
\]

Choose \( P \in g(V)(L) \), say \( P = (x_0, \ldots, x_m) \) with \( x_i \in L \). It is easy to see that

\[
\| x_{n+i} \|^q_w \leq c_w \cdot \max_{j \leq m} \| x_j \|^{q-1}_w \cdot \max_{j \leq n} \| x_j \|_w,
\]

for \( i = 0, \ldots, m - n \). Put

\[
c'_w = \max \{ 1, c^q_w \},
\]

then

\[
\max_{i \leq m} \| x_i \|_w \leq c'_w \cdot \max_{j \leq n} \| x_j \|_w.
\]

In particular,

\[
H_L(x_0; \cdots; x_m) = \prod_{w \in M_L} \max_{i \leq m} \| x_i \|_w
\]

\[
\leq \left( \prod_{w \in M_L} c'_w \right) \left( \prod_{w' \in M_L} \max_{j \leq n} \| x_j \|_w \right)^d
\]

\[
= \left( \prod_{w \in M_K} c'_w \right) H_L(x_0; \cdots; x_n),
\]

4
where \( d = [L: K] \). Therefore
\[
h(x_0; \ldots; x_m) \leq h(x_0; \ldots; x_n) + c,
\]
where \( c = \frac{1}{[K: Q]} \sum_{\iota \in \mathcal{M}_K} c'_{\iota} \) which neither depends on \( P \) nor on \( L \). Hence \( h_g - h_f \) is bounded from above.

As a consequence, we can define, up to equivalence, a height function \( h_{\mathcal{L}} \) for every invertible sheaf \( \mathcal{L} \) on \( V \) which is basepoint-free. For, choose a morphism \( f \) over \( K \) from \( V \) into \( \mathbb{P}^n \) such that
\[
\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}^n}(1).
\]
(Such a morphism exists since \( \mathcal{L} \) is basepoint-free.) Then, by Theorem 3,
\[
h_{\mathcal{L}} = h_f
\]
depends only, up to equivalence, on \( \mathcal{L} \). More precisely, one defines \( h_{\mathcal{L}} \) as the equivalence class of \( h_f \). That is, if \( \mathcal{H}(V(K)) \) is the group of equivalence classes of real-valued functions on \( V(K) \), we have \( h_{\mathcal{L}} \in \mathcal{H}(V(K)) \). However, often we will treat \( h_{\mathcal{L}} \) as a real-valued function, keeping in mind that \( h_{\mathcal{L}} \) is only defined up to equivalence. It is easy to prove that, for any basepoint-free invertible sheaves \( \mathcal{L} \) and \( \mathcal{M} \) on \( V \),
\[
h_{\mathcal{L} \otimes \mathcal{M}} \sim h_{\mathcal{L}} + h_{\mathcal{M}}.
\]
As a consequence, for any invertible sheaf \( \mathcal{L} \) on \( V \), we can define, up to equivalence, a height function \( h_{\mathcal{L}} \) by
\[
h_{\mathcal{L}} = h_{\mathcal{L}_1} - h_{\mathcal{L}_2},
\]
where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are basepoint-free invertible sheaves on \( V \) such that
\[
\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}.
\]
(Such sheaves always exist; see [HAG], p. 121.) By (1), this does not depend on \( \mathcal{L}_1, \mathcal{L}_2 \). Hence the following result due to A. Weil.

**Theorem 4** Let \( V \) be a projective algebraic variety over \( K \). There exists a unique homomorphism
\[
h: \text{Pic} \, V \rightarrow \mathcal{H}(V(K))
\]
such that
(i) if $V = \mathbb{P}^n$ then $h_{\mathcal{O}_{\mathbb{P}^n}}(1)$ is the usual height $h$ on projective space.
(ii) if $W$ is a projective algebraic variety over $K$ and $f: V \to W$ is a morphism over $K$ then

\[ h_{f^*\mathcal{L}} = h_{\mathcal{L}} \circ f, \]

for any $\mathcal{L} \in \text{Pic } W$.

It is then easy to prove, using Theorem 2, the following finiteness theorem.

**Theorem 5** Let $V$ be a projective algebraic variety over $K$. If $\mathcal{L}$ is an ample sheaf on $V$ then, for all constants $C$ and $d$, the set

\[ \{ P \in V(K) \mid h_{\mathcal{L}}(P) \leq C[K(P) : K] \leq d \} \]

is a finite set.

Observe that it makes sense to call an element of $\mathcal{H}(V)$ bounded from below (or above).

**Theorem 6** Let $V$ be a projective algebraic variety over $K$. If $\mathcal{L}$ is an invertible sheaf on $V$ and $s$ is a global section then $h_{\mathcal{L}}$ is bounded from below on the set

\[ \{ P \in V(K) \mid s(P) \neq 0 \}. \]

**Proof.** Choose basepoint-free invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ on $V$ such that

\[ \mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}. \]

Let $s_0, \ldots, s_n$ be global sections of $\mathcal{L}_2$ that generate $\mathcal{L}_2$. Choose global sections $s_{n+1}, \ldots, s_m$ of $\mathcal{L}_1$ such that

\[ s \otimes s_0, \ldots, s \otimes s_n, s_{n+1}, \ldots, s_m \]

generate $\mathcal{L}_1$. Then, whenever $P \in V(K)$ with $s(P) \neq 0$,

\[
\begin{align*}
    h_{\mathcal{L}_1}(P) &= h(s \otimes s_0(P), \ldots, s \otimes s_n(P), s_{n+1}(P), \ldots, s_m(P)) \\
    &\geq h(s \otimes s_0(P), \ldots, s \otimes s_n(P)) \\
    &= h(s_0(P), \ldots, s_n(P)) \\
    &= h_{\mathcal{L}_2}(P).
\end{align*}
\]

Therefore, $h_{\mathcal{L}}$ is bounded from below on the set of $P \in V(K)$ such that $s(P) \neq 0$. \qed
3 Heights on abelian varieties

We will need the Theorem of the Cube.

**Theorem 7** Let $X_1, X_2, X_3$ be complete algebraic varieties over the field $K$ and let $P_i \in X_i(K)$. Then, an invertible sheaf $\mathcal{L}$ on $X_1 \times X_2 \times X_3$ is trivial whenever its restrictions to $\{P_1\} \times X_2 \times X_3$, $X_1 \times \{P_2\} \times X_3$ and $X_1 \times X_2 \times \{P_3\}$ are trivial.

**Proof.** Let us give a proof when $\text{char}(K) = 0$, since this is the case we are interested in. Then it suffices to prove the theorem for $K = \mathbb{C}$.

Before we continue the proof let us recall the following definition. A contravariant functor $F$ from the category of complete complex algebraic varieties with basepoints into the category of abelian groups is called of order $\leq n$ if for all complete complex algebraic varieties $X_0, \ldots, X_n$ with basepoints, the natural mapping

$$F\left(\prod_{i=0}^{n} X_i\right) \longrightarrow \prod_{j=0}^{n} F\left(\prod_{i \neq j} X_i\right)$$

is an isomorphism. As an example, the Theorem of the Cube states that the functor $\text{Pic}$ is of order $\leq 2$.

To finish the proof we switch to an analytic point of view. Let $\mathcal{O}_{X, h}$ denote the sheaf of analytic functions on $X$. According to the GAGA-principle,

$$\text{Pic} X \cong H^1(X, \mathcal{O}_{X, h}^*),$$

for any complete complex algebraic variety $X$. The long exact sequence associated to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X, h} \xrightarrow{\text{exp}} \mathcal{O}_{X, h}^* \longrightarrow 0$$

implies the existence of an exact sequence

$$H^1(X, \mathcal{O}_{X, h}) \longrightarrow H^1(X, \mathcal{O}_{X, h}^*) \longrightarrow H^2(X, \mathbb{Z}).$$

Since both $H^1(\cdot, \mathcal{O}_{X, h})$ and $H^2(\cdot, \mathbb{Z})$ are functors of order $\leq 2$, the functor $H^1(\cdot, \mathcal{O}_{X, h}^*)$ is of order $\leq 2$. This proves the theorem. \qed
Corollary 8 Let $X$ be an abelian variety over the field $K$ and let $p_i: X^3 \to X$ be the projection on the $i$th factor. Let $p_{ij} = p_i + p_j$ and $p_{ijk} = p_i + p_j + p_k$. Then, for any invertible sheaf $\mathcal{L}$ on $X$, the invertible sheaf
\[
p_{123}^* \mathcal{L} \otimes p_{12}^* \mathcal{L}^{-1} \otimes p_{13}^* \mathcal{L}^{-1} \otimes p_{23}^* \mathcal{L}^{-1} \otimes p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L}
\]
on $X^3$ is trivial.

Proof. Taking restrictions of this sheaf to $O \times X \times X$, $X \times O \times X$ and $X \times X \times O$ yields a trivial sheaf. The conclusion follows from the Theorem of the Cube. \qed

Since this corollary expresses a relation between sheaves on $X^3$, we have immediately, by Theorem 4, the following fact about heights on abelian varieties.

Theorem 9 If $X$ is an abelian variety over a number field $K$ then, for any invertible sheaf $\mathcal{L}$ on $X$,
\[
h_\mathcal{L}(P+Q+R) - h_\mathcal{L}(P+Q) - h_\mathcal{L}(P+R) - h_\mathcal{L}(Q+R) + h_\mathcal{L}(P) + h_\mathcal{L}(Q) + h_\mathcal{L}(R) \sim 0,
\]
as functions on $X(\overline{K})^3$. 

Let us denote for an abelian variety $X$ over $K$ the multiplication-by-$n$ mapping from $X$ into itself by $[n]$, for any integer $n$. Recall that an invertible sheaf $\mathcal{L}$ is called symmetric (resp. antisymmetric) whenever $[-1]^* \mathcal{L} \cong \mathcal{L}$ (resp. $[-1]^* \mathcal{L} \cong \mathcal{L}^{-1}$). As a consequence of Corollary 8, one can prove the following.

Corollary 10 If $X$ is an abelian variety over the field $K$ and $\mathcal{L}$ is an invertible sheaf on $X$ then
\[
[n]^* \mathcal{L} \cong \mathcal{L}^{(n^2+n)/2} \otimes [-1]^* \mathcal{L}^{(n^2-n)/2},
\]
for any integer $n$. In particular,
\[
[n]^* \mathcal{L} \cong \begin{cases} 
\mathcal{L}^{n^2}, & \text{if } \mathcal{L} \text{ is symmetric}, \\
\mathcal{L}^n, & \text{if } \mathcal{L} \text{ is antisymmetric}.
\end{cases}
\]
Again, this translates into properties of heights on abelian varieties.

**Theorem 11** If \( X \) is an abelian variety over the number field \( K \) and \( \mathcal{L} \) is an invertible sheaf on \( X \) then

\[
h_\mathcal{L} \circ [n] \sim \frac{n^2 + n}{2} h_\mathcal{L} + \frac{n^2 - n}{2} h_\mathcal{L} \circ [-1],
\]

for any integer \( n \). In particular,

\[
h_\mathcal{L} \circ [n] \sim \begin{cases} 
  n^2 h_\mathcal{L}, & \text{if } \mathcal{L} \text{ is symmetric,} \\
  nh_\mathcal{L}, & \text{if } \mathcal{L} \text{ is antisymmetric.}
\end{cases}
\]

The property of \( h_\mathcal{L} \) in Theorem 9 will imply the existence of a canonical height function, the *Néron-Tate height relative to \( \mathcal{L} \),

\[
\hat{h}_\mathcal{L}: X(\overline{K}) \rightarrow \mathbb{R},
\]
as stated in the following theorem.

**Theorem 12** If \( X \) is an abelian variety over the number field \( K \) and \( \mathcal{L} \) is an invertible sheaf on \( X \) then there exist a unique symmetric bilinear mapping \( b_\mathcal{L}: X(\overline{K}) \times X(\overline{K}) \rightarrow \mathbb{R} \) and a unique linear mapping \( l_\mathcal{L}: X(\overline{K}) \rightarrow \mathbb{R} \), such that

\[
\hat{h}_\mathcal{L}: X(\overline{K}) \rightarrow \mathbb{R},
\]
defined by

\[
\hat{h}_\mathcal{L}(P) = \frac{1}{2} b_\mathcal{L}(P, P) + l_\mathcal{L}(P),
\]
is equivalent to \( h_\mathcal{L} \). Moreover, if \( \mathcal{L} \) is symmetric then \( l_\mathcal{L} = 0 \) and if \( \mathcal{L} \) is ample then \( b_\mathcal{L} \) is positive definite on \( X(\overline{K}) \otimes \mathbb{R} \).

**Proof.** The existence and uniqueness of \( b_\mathcal{L} \) and \( l_\mathcal{L} \) follow from the lemma below, whose proof is left to the reader.

If \( \mathcal{L} \) is symmetric then in virtue of Theorem 11

\[
\frac{1}{2} n^2 b_\mathcal{L}(P, P) + nl_\mathcal{L}(P) = \hat{h}_\mathcal{L}(nP) = n^2 \hat{h}_\mathcal{L}(P) = \frac{1}{2} n^2 b_\mathcal{L}(P, P) + n^2 l_\mathcal{L}(P),
\]

for any integer \( n \). Hence \( l_\mathcal{L} = 0 \).

If \( \mathcal{L} \) is ample then \([-1]^* \mathcal{L} \) is ample too. Hence, \( \mathcal{M} = \mathcal{L} \otimes [-1]^* \mathcal{L} \) is ample. Moreover by uniqueness

\[
b_\mathcal{L} = \frac{1}{2} b_\mathcal{L} + \frac{1}{2} b_{[-1]^* \mathcal{L}} = \frac{1}{2} b_\mathcal{M}.
\]
Therefore it suffices to prove that $b_\mathcal{L}$ is positive definite on $X(K) \otimes \mathbb{R}$ for any symmetric ample invertible sheaf $\mathcal{L}$.

Since then $\hat{h}_\mathcal{L} = \frac{1}{2}b_\mathcal{L}$, it follows from Theorem 5 that for any finitely generated subgroup $A$ of $X(K)$ and for any $C \in \mathbb{R}$ the cardinality of the set

$$\{ P \in A \mid b_\mathcal{L}(P, P) \leq C \}$$

is finite. It is not difficult to prove that this implies that $b_\mathcal{L}$ is positive definite on $X(K) \otimes \mathbb{R}$. □

**Lemma 13** Let $G$ be an abelian group and $h: G \to \mathbb{R}$ a function such that $h(P + Q + R) = h(P + Q) + h(P + R) - h(Q + R) + h(P) + h(Q) + h(R) \sim 0$, as functions on $G^3$. Then there exists a unique symmetric bilinear mapping $b: G \times G \to \mathbb{R}$ and a unique homomorphism $l: G \to \mathbb{R}$ such that $h \sim \hat{h}$, where

$$\hat{h}(P) = \frac{1}{2}b(P, P) + l(P).$$

## 4 Metrized line bundles

In this section $K$ will be a number field, $R$ its ring of integers and $X$ a projective scheme over $R$.

If $\mathcal{L}$ is a line bundle (more precisely, an invertible sheaf) on $\text{Spec } R$ the $R$-module $\Gamma(\text{Spec } R, \mathcal{L})$ is projective of rank 1. A metrized line bundle on $\text{Spec } R$ is a line bundle $\mathcal{L}$ together with $v$-adic metrics $\| \cdot \|_v$ on $\Gamma(\text{Spec } R, \mathcal{L}) \otimes_R K_v$, for any infinite place $v$ of $K$. As an example, the trivial sheaf $\mathcal{R}$ together with the standard norms $\| \cdot \|_v$ on $K_v$ is a metrized line bundle on $\text{Spec } R$. If $\mathcal{L}$ is a metrized line bundle on $\text{Spec } R$ then the degree of $\mathcal{L}$ is the real number

$$\deg \mathcal{L} = \log \#(M/\mathbb{R}s) - \sum_{v \in M_{\mathbb{R}}} \log \|s\|_v,$$

for any nonzero $s \in M = \Gamma(\text{Spec } R, \mathcal{L})$. (It is easy to check that the right-hand side is independent of $s$.)

If $\mathcal{L}$ is an invertible sheaf on $X$ and $v$ is a place of $K$ then $\mathcal{L}$ is the sheaf of sections of some geometrical line bundle on $X$ whose set of $K_v$-rational points will be denoted by $\mathcal{L}(K_v)$. Observe that $\mathcal{L}(K_v)$ is a topological line
bundle on $X(K_v)$. A $v$-adic metric on $\mathcal{L}$ is a continuously varying $v$-adic metric on each fibre of $\mathcal{L}(K_v)$. A metrized line bundle on $X$ is a line bundle $\mathcal{L}$ on $X$ together with $v$-adic metrics on $\mathcal{L}$, for every $v \in \mathbb{M}_K^\infty$.

**Example 14.** Let $X = \mathbb{P}^n_R$ and $\mathcal{L} = \mathcal{O}(d)$, with $d \geq 0$. Then $\mathcal{L}$ is generated by global sections and $\Gamma(X, \mathcal{L})$ is just the $R$-module of homogeneous polynomials in $x_0, \ldots, x_n$ over $R$ of degree $d$. If $v$ is an infinite place of $K$ then, for any $f \in \Gamma(X, \mathcal{L})$,

$$X(K_v) \longrightarrow \mathbb{R}$$

$$P \longmapsto \frac{\|f(P)\|_v}{\max_{i \leq n} \|x_i(P)\|_v^d}$$

defines a $v$-adic metric on $\mathcal{L}$. □

**Lemma 15** Let $\mathcal{L}$ be a line bundle on $X$. If $v \in \mathbb{M}_K^\infty$ and $\| \cdot \|_v$ and $\| \cdot \|'_v$ are $v$-adic metrics on $\mathcal{L}$ then there exist $c_1, c_2 > 0$ such that

$$c_1 \| \cdot \|_v \leq \| \cdot \|'_v \leq c_2 \| \cdot \|_v$$

on $\mathcal{L}(K_v)$.

**Proof.** Follows from the fact that $X(K_v)$ is compact. □

If $\mathcal{L}$ is a metrized line bundle on $X$ and $P$ is an $R$-rational point of $X$, that is, $P$ is a morphism of $R$-schemes

$$\text{Spec } R \longrightarrow X,$$

then $P^*\mathcal{L}$ is a metrized line bundle on $\text{Spec } R$. Observe that $P_K$ is a $K$-rational point. We will prove the following theorem using Lemma 15.

**Theorem 16** If $\mathcal{L}$ is a metrized line bundle on $X$ then

$$\deg P^*\mathcal{L} \sim [K : \mathbb{Q}] h_\mathcal{L}(P_K)$$

as real-valued functions on $X(R)$. 

11
Proof. It suffices to prove the theorem for $\mathcal{L} \cong \mathcal{O}(1)$. In virtue of Lemma 15 we may choose a convenient metric on $\mathcal{L}$. Let us define, for any infinite place $v$ of $K$, the $v$-adic metric $\| \cdot \|_v$ on $\mathcal{O}(1)$ as in Example 14.

We will compute $\deg P^*\mathcal{O}(1)$, where $P$ is an $R$-rational point of $X \subseteq \mathbb{P}^n_R$. We may assume that $P^*x_0$ is a nonzero section of $\Gamma(\text{Spec } R, P^*\mathcal{O}(1))$. Then,

$$
P^*\mathcal{O}(1)/RP^*x_0 \cong (\sum Rx_i(P))/Rx_0(P) \cong (\sum R\frac{x_i}{x_0}(P))/R.
$$

Hence,

$$
\#P^*\mathcal{O}(1)/RP^*x_0 = \prod_{v \notin M_K^\infty} \max_{i \leq n} \| \frac{x_i}{x_0}(P) \|_v
$$

$$
= \left( \prod_{v \notin M_K^\infty} \max_{i \leq n} \| x_i(P) \|_v \right) \cdot \prod_{v \in M_K^\infty} \| x_0(P) \|_v.
$$

Therefore,

$$
\deg P^*\mathcal{O}(1) = \sum_{v \notin M_K^\infty} \log \max_{i \leq n} \| x_i(P) \|_v + \sum_{v \in M_K^\infty} \log \| x_0(P) \|_v +
$$

$$
- \sum_{v \in M_K^\infty} \log \frac{\| x_0(P) \|_v}{\max_{i \leq n} \| x_i(P) \|_v}
$$

$$
= [K: \mathbb{Q}] h_{\mathcal{L}}(P_K)
$$

$\square$