THE GROUP OF ALGEBRAIC DIFFEOMORPHISMS OF A REAL RATIONAL SURFACE IS $n$-TRANSITIVE

JOHANNES HUISMAN AND FRÉDÉRIC MANGOLTE

Abstract. Let $X$ be a rational nonsingular compact connected real algebraic surface. Denote by $\text{Diff}_{\text{alg}}(X)$ the group of algebraic diffeomorphisms of $X$ into itself. The group $\text{Diff}_{\text{alg}}(X)$ acts diagonally on $X^n$, for any natural integer $n$. We show that this action is transitive, for all $n$.

As an application we give a new and simpler proof of the fact that two rational nonsingular compact connected real algebraic surfaces are algebraically diffeomorphic if and only if they are homeomorphic as topological surfaces.

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1. Introduction

Let $X$ be a nonsingular compact connected real algebraic manifold, i.e., $X$ is a compact connected submanifold of $\mathbb{R}^n$ defined by real polynomial equations, where $n$ is some natural integer. We study the group of algebraic diffeomorphisms of $X$ into itself. Let us make precise what we mean by an algebraic diffeomorphism.

An algebraic map $\varphi$ of $X$ into itself is a map of the form

$$\varphi(x) = \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_n(x)}{q_n(x)}\right)$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n$ are real polynomials in the variables $x_1, \ldots, x_n$, with $q_i(x) \neq 0$ for any $x \in X$. An algebraic map from $X$ into itself is also called a regular map [2]. Note that an algebraic map is necessarily of class $C^\infty$. An algebraic map $\varphi$ is an algebraic diffeomorphism if $\varphi$ is algebraic, bijective and $\varphi^{-1}$ is algebraic. An algebraic diffeomorphism from $X$ into itself is also called a biregular map [2]. We denote by $\text{Diff}_{\text{alg}}(X)$ the group of algebraic diffeomorphism of $X$ into itself.

For a general real algebraic manifold, the group $\text{Diff}_{\text{alg}}(X)$ tends to be rather small. For example, if $X$ admits a complexification $\mathcal{X}$ of general type then $\text{Diff}_{\text{alg}}(X)$ is finite. Indeed, any algebraic diffeomorphism of $X$ into itself is the restriction to $X$ of a birational automorphism of $\mathcal{X}$. The group of birational automorphisms of $\mathcal{X}$ is known to be finite [7]. Therefore, $\text{Diff}_{\text{alg}}(X)$ is finite for such real algebraic manifolds.

In the current paper, we study the group $\text{Diff}_{\text{alg}}(X)$ when $X$ is a compact connected real algebraic surface, i.e., a compact connected real algebraic

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manifold of dimension 2. By what is said above, the group of algebraic diffeomorphisms of such a surface is most interesting when the Kodaira dimension of $X$ is equal to $-\infty$, and, in particular, when $X$ is geometrically rational. By a result of Comessatti, a connected geometrically rational real surface is rational (see Theorem IV of [4] and the remarks thereafter, or [8, Corollary VI.6.5]). Therefore, we will concentrate our attention to the group $\text{Diff}_\text{alg}(X)$ when $X$ is a rational compact connected real algebraic surface.

Recall that a real algebraic surface $X$ is rational if there are a nonempty Zariski open subset $U$ of $\mathbb{R}^2$, and a nonempty Zariski open subset $V$ of $X$, such that $U$ and $V$ are algebraically diffeomorphic. In particular, this means that $X$ contains a nonempty Zariski open subset $V$ that admits a parametrization by real rational functions in two variables.

Examples of rational real algebraic surfaces are the following:

- the unit sphere $S^2$ defined by the equation $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$,
- the real algebraic torus $S^1 \times S^1$, where $S^1$ is the unit circle defined by the equation $x^2 + y^2 = 1$ in $\mathbb{R}^2$,
- the real projective plane $\mathbb{P}^2(\mathbb{R})$ (refer to [2, Theorem 3.4.4] for an explicit realization of $\mathbb{P}^2(\mathbb{R})$ as a real algebraic surface), and
- any real algebraic surface obtained from one of the above ones by repeatedly blowing up a real point.

The following conjecture has attracted our attention.

**Conjecture 1.2** ([1, Conjecture 1.4]). Let $X$ be a rational nonsingular compact connected real algebraic surface. Let $n$ be a natural integer. Then the group $\text{Diff}_\text{alg}(X)$ acts $n$-transitively on $X$.

The conjecture seems known to be true only in the case where $X$ is algebraically diffeomorphic to $S^1 \times S^1$:

**Theorem 1.3** ([1, Theorem 1.3]). The group $\text{Diff}_\text{alg}(S^1 \times S^1)$ acts $n$-transitively on $S^1 \times S^1$, for any natural integer $n$. \[\square\]

The object of the paper is to prove Conjecture 1.2 for all rational surfaces:

**Theorem 1.4.** The group $\text{Diff}_\text{alg}(X)$ acts $n$-transitively on $X$, whenever $X$ is a rational nonsingular compact connected real algebraic surface, and $n$ is a natural integer.

Our proof goes as follows. We first prove $n$-transitivity of $\text{Diff}_\text{alg}(S^2)$ (see Theorem 2.3). For this, we need a large class of algebraic diffeomorphisms of $S^2$ into itself. Lemma 2.1 constructs such a large class. Once $n$-transitivity of $\text{Diff}_\text{alg}(S^2)$ is established, we prove $n$-transitivity of $\text{Diff}_\text{alg}(X)$, for any other rational surface $X$, by the following argument.

If $X$ is algebraically diffeomorphic to $S^1 \times S^1$ then the $n$-transitivity has been proved in [1, Theorem 1.3]. Therefore, we may assume that $X$ is not algebraically diffeomorphic to $S^1 \times S^1$. It follows from the Minimal Model Program for real algebraic surfaces, due to János Kollár [5, 6], that $X$ is isomorphic to a blowing-up of $S^2$ in $m$ points, for some natural integer $m$ (see Theorems 4.1 and 4.3 for precise statements). The $n$-transitivity of $\text{Diff}_\text{alg}(X)$ will then be proved by induction on $m$. 
Lemma 2.2. Let \( f \) and let \( \alpha \) of \( S \)

Theorem 1.5 shows that the group of algebraic diffeomorphisms of a rational real algebraic surface is big. It would, therefore, be particularly interesting to study the dynamics of algebraic diffeomorphisms of rational real surfaces, as is done for K3-surfaces in [3], for example.

As an application of Theorem 1.4, we present in Section 5 a simplified proof of the following result.

**Theorem 1.5 ([1, Theorem 1.2]).** Let \( X \) and \( Y \) be rational nonsingular compact connected real algebraic surfaces. Then the following statements are equivalent.

1. The real algebraic surfaces \( X \) and \( Y \) are algebraically diffeomorphic.
2. The topological surfaces \( X \) and \( Y \) are homeomorphic.

Indeed, the Minimal Model Program for real algebraic surfaces and the \( n \)-transitivity of \( \text{Diff}_{\text{alg}}(\mathbb{P}^2(\mathbb{R})) \) suffice to deduce that result (see the remark following Theorem 1.2 of [1]).

2. \( n \)-Transitivity of \( \text{Diff}_{\text{alg}}(S^2) \)

We need to slightly extend the notion of an algebraic map between real algebraic manifolds. Let \( X \) and \( Y \) be real algebraic submanifolds of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively. Let \( A \) be any subset of \( X \). An algebraic map from \( A \) into \( Y \) is a map \( \varphi \) as in (1.1), where \( p_1, \ldots, p_n, q_1, \ldots, q_n \) are real polynomials in the variables \( x_1, \ldots, x_m \), with \( q_i(x) \neq 0 \) for any \( x \in A \). To put it otherwise, a map \( \varphi \) from \( A \) into \( Y \) is algebraic if there is a Zariski open subset \( U \) of \( X \) containing \( A \) such that \( \varphi \) is the restriction of an algebraic map from \( U \) into \( Y \).

We will consider algebraic maps from a subset \( A \) of \( X \) into \( Y \), in the special case where \( X \) is algebraically diffeomorphic to the real algebraic line \( \mathbb{R} \), the subset \( A \) of \( X \) is a closed interval, and \( Y \) is algebraically diffeomorphic to the real algebraic group \( \text{SO}_2(\mathbb{R}) \).

Denote by \( S^2 \) the 2-dimensional sphere defined in \( \mathbb{R}^3 \) by \( x^2 + y^2 + z^2 = 1 \).

**Lemma 2.1.** Let \( L \) be a line through the origin of \( \mathbb{R}^3 \) and denote by \( I \subset L \) the closed interval whose boundary is \( L \cap S^2 \). Denote by \( L^\perp \) the plane orthogonal to \( L \) containing the origin. Let \( f: I \to \text{SO}(L^\perp) \) be an algebraic map. Define \( \varphi_f: S^2 \to S^2 \) by

\[
\varphi_f(z,x) = (f(x)z,x)
\]

where \((z,x) \in (L^\perp \oplus L) \cap S^2 \). Then \( \varphi_f \) is an algebraic diffeomorphism of \( S^2 \).

**Proof.** Identifying \( \mathbb{R}^2 \) with \( \mathbb{C} \), we may assume that \( S^2 \subset \mathbb{C} \times \mathbb{R} \) is given by the equation \( |z|^2 + x^2 = 1 \), and \( L \) is the line \( \{0\} \times \mathbb{R} \). Then \( L^\perp = \mathbb{C} \times \{0\} \) and \( \text{SO}(L^\perp) = S^1 \). The map \( \varphi_f \) is an algebraic map from \( S^2 \) into itself. Let \( \overline{f} \) be the complex conjugate of \( f \), i.e. \( \forall x \in I, \overline{f}(x) = \overline{f(x)} \). We have \( \varphi_{\overline{f}} \circ \varphi_f = \varphi_f \circ \varphi_{\overline{f}} = \text{id} \). Therefore \( \varphi_f \) is an algebraic diffeomorphism of \( S^2 \).

**Lemma 2.2.** Let \( x_1, \ldots, x_n \) be \( n \) distinct points of the closed interval \([-1,1] \), and let \( \alpha_1, \ldots, \alpha_n \) be elements of \( \text{SO}_2(\mathbb{R}) \). Then there is an algebraic map \( f: [-1,1] \to \text{SO}_2(\mathbb{R}) \) such that \( f(x_j) = \alpha_j \) for \( j = 1, \ldots, n \).
Proof. Since $SO_2(\mathbb{R})$ is algebraically diffeomorphic to the unit circle $S^1$, it suffices to prove the statement for $S^1$ instead of $SO_2(\mathbb{R})$. Let $P$ be a point of $S^1$ distinct from $\alpha_1, \ldots, \alpha_n$. Since $S^1 \setminus \{P\}$ is algebraically diffeomorphic to $\mathbb{R}$, it suffices, finally, to prove the statement for $\mathbb{R}$ instead of $SO_2(\mathbb{R})$. The latter statement is an easy consequence of Lagrange polynomial interpolation. \qed

Theorem 2.3. Let $n$ be a natural integer. The group $\text{Diff}_{\text{alg}}(S^2)$ acts transitively on $S^2$.

Proof. We will need the following terminology. Let $W$ be a point of $S^2$, let $L$ be the line in $\mathbb{R}^3$ passing through $W$ and the origin. The intersection of $S^2$ with any plane in $\mathbb{R}^3$ that is orthogonal to $L$ is called a parallel of $S^2$ with respect to $W$.

Let $P_1, \ldots, P_n$ be $n$ distinct points of $S^2$, and let $Q_1, \ldots, Q_n$ be $n$ distinct points of $S^2$. We need to show that there is an algebraic diffeomorphism $\varphi$ from $S^2$ into itself such that $\varphi(P_j) = Q_j$, for all $j$.

Up to a projective linear automorphism of $\mathbb{P}^3(\mathbb{R})$ fixing $S^2$, we may assume that all the points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ are in a sufficiently small neighborhood of the north pole $N := (0, 0, 1)$ of $S^2$. Indeed, we may assume that none of the points is contained in a small spherical disk $D$ centered at $N$. Then the images of the points by the inversion with respect to the boundary of $D$ are contained in $D$.

We can choose two points $W$ and $W'$ of $S^2$ in the $xy$-plane such that the angle $WOW'$ is equal to $\pi/2$ and such that the following property holds. Any parallel with respect to $W$ contains at most one of the points $P_1, \ldots, P_n$, and any parallel with respect to $W'$ contains at most one of $Q_1, \ldots, Q_n$. Denote by $\Gamma_j$ the parallel with respect to $W$ that contains $P_j$, and by $\Gamma'_j$ the one with respect to $W'$ that contains $Q_j$.

Since the disk $D$ has been chosen sufficiently small, $\Gamma_j \cap \Gamma'_j$ is nonempty for all $j = 1, \ldots, n$. Let $R_j$ be one of the intersection points of $\Gamma_j$ and $\Gamma'_j$ (see Figure 1). It is now sufficient to show that there is an algebraic diffeomorphism $\varphi$ of $S^2$ such that $\varphi(P_j) = R_j$.

Let again $L$ be the line in $\mathbb{R}^3$ passing through $W$ and the origin. Denote by $I \subset L$ the closed interval whose boundary is $L \cap S^2$. Let $x_j$ be the unique element of $I$ such that $\Gamma_j = (x_j + L^\perp) \cap S^2$. Let $\alpha_j \in SO(L^\perp)$ be such that $\alpha_j(P_j - x_j) = R_j - x_j$. According to Lemma 2.2, there is an algebraic map $f: I \to SO(L^\perp)$ such that $f(x_j) = \alpha_j$. Let $\varphi := \varphi_f$ as in Lemma 2.1. By construction, $\varphi(P_j) = R_j$, for all $j = 1, \ldots, n$. \qed

3. Contractible curves

Let $Y$ be a real algebraic surface and let $P$ be a nonsingular point of $Y$. We denote by $B_P(Y)$ the blow-up of $Y$ at $P$.

Definition 3.1. Let $X$ be a projective real algebraic surface. Let $C$ be a real algebraic curve contained in $X$. We say that $C$ is contractible if there is a projective real algebraic surface $Y$, a nonsingular point $P \in Y$, and an algebraic diffeomorphism $\varphi: B_P(Y) \to X$ such that $\varphi^{-1}(C)$ is equal to the exceptional curve of $B_P(Y)$ over $P$. By abuse of language, we will then also say that $Y$ is obtained from $X$ by contracting $C$ to a point.
If a curve $C$ is contractible, then $C$ is nonsingular, irreducible and rational. Moreover, $C$ is contained in the set of nonsingular points of $X$. In this paper, we will only consider contractible curves in nonsingular surfaces.

**Theorem 3.2.** Let $X$ be a nonsingular projective real algebraic surface. Let $C$ be a nonsingular rational irreducible real algebraic curve contained in $X$. Assume that

1. $X$ admits a nonsingular projective complexification $\mathcal{X}$ in which the Zariski closure $\mathcal{C}$ of $C$ is nonsingular and such that the self-intersection $\mathcal{C}^2$ is greater than or equal to $-1$, and
2. the normal bundle of $C$ in $X$ is nontrivial.

Then $C$ is contractible. Moreover, the surface $Y$ obtained from $X$ by contracting $C$ to a point is nonsingular.

**Proof.** Let $\mathcal{X}$ be a nonsingular projective complexification of $X$ such that the Zariski closure $\mathcal{C}$ of $C$ in $\mathcal{X}$ is nonsingular and $\mathcal{C}^2 \geq -1$. Since $C$ is rational, $C$ is diffeomorphic to a circle. Since the normal bundle of $C$ in $X$ is nontrivial, the degree of $\mathcal{I}|_{\mathcal{C}}$ is odd, where $\mathcal{I}$ is the ideal sheaf of $\mathcal{C}$ in $\mathcal{X}$. It follows that the self-intersection of $\mathcal{C}$ is odd. Let $k$ be an integer such that $\mathcal{C}^2 = 2k - 1$. Since $\mathcal{C}^2 \geq -1$, one has $k \geq 0$. On $\mathcal{C}$, choose $k$ pairs of complex conjugate points $P_1, Q_1, \ldots, P_{2k}, Q_{2k}$. Let $\tilde{\mathcal{X}}$ be the blow-up of $\mathcal{X}$ at these points. The surface $\tilde{\mathcal{X}}$ is again a nonsingular projective complexification of $X$. Moreover, the strict transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ in $\tilde{\mathcal{X}}$ is a nonsingular rational algebraic curve defined over $\mathbb{R}$ whose self-intersection is equal to $-1$. Then there is a nonsingular projective algebraic surface $\mathcal{Y}$ defined over $\mathbb{R}$, a nonsingular real point $P \in \mathcal{Y}$, and an isomorphism $\Phi: B_P(\mathcal{Y}) \to \tilde{\mathcal{X}}$ such that $\Phi^{-1}(\tilde{\mathcal{C}})$ is equal to the exceptional curve of $B_P(\mathcal{Y})$ over $P$. To put it otherwise, the surface $\mathcal{Y}$ defined over $\mathbb{R}$ is obtained from $\tilde{\mathcal{X}}$ by contracting $\mathcal{C}$ to a point. It follows that the set of real points $Y$ of
$Y$ is a nonsingular projective real algebraic surface obtained from $X$ by contracting $C$ to point. It is clear that $Y$ is nonsingular. □

4. $n$-Transitivity of $\text{Diff}_{\text{alg}}(X)$

We reformulate a result of [1] and adapt it to our purposes:

**Theorem 4.1** ([1, Theorem 3.1]). Let $X$ be a rational nonsingular compact connected real algebraic surface. Then,

1. $X$ is either algebraically diffeomorphic to $S^1 \times S^1$, or
2. $X$ is algebraically diffeomorphic to a real algebraic surface obtained from $S^2$ by successively blowing up. □

It is in 4.1 that Kollár’s Minimal Model Program for real algebraic surfaces is used.

If $X$ is a rational surface algebraically diffeomorphic to a successive blowing-up of $S^2$, as in Theorem 4.1 above, then one can get rid of the adjective "successive" by using the following statement (compare [1, Lemma 4.1] and how it is used to prove [1, Lemma 4.3]).

**Lemma 4.2.** Let $P \in S^2$ and let $C \subseteq S^2$ be an euclidean circle in $S^2$ containing $P$. Let $B_P(S^2)$ be the blowing-up of $S^2$ at $P$, and let $E$ be the exceptional curve of $B_P(S^2)$ over $P$. Denote by $\tilde{C} \subset B_P(S^2)$ the strict transform of $C$. Then there is an algebraic diffeomorphism $\varphi$ of $B_P(S^2)$ into itself such that $\varphi(E) = \tilde{C}$.

Proof. The statement immediately follows from the fact that $B_P(S^2)$ is algebraically diffeomorphic to the real projective plane $\mathbb{P}^2(\mathbb{R})$, and that $E$ and $\tilde{C}$ are real projective lines on $\mathbb{P}^2(\mathbb{R})$. □

The following sharpened version of Theorem 4.1 follows:

**Theorem 4.3.** Let $X$ be a rational nonsingular compact connected real algebraic surface. Then,

1. $X$ is either algebraically diffeomorphic to $S^1 \times S^1$, or
2. there are distinct points $R_1, \ldots, R_m$ of $S^2$ such that $X$ is algebraically diffeomorphic to the real algebraic surface obtained from $S^2$ by blowing up $R_1, \ldots, R_m$. □

**Proof of Theorem 1.4.** Let $X$ be a rational surface. By Theorem 4.3, $X$ is algebraically diffeomorphic to $S^1 \times S^1$ or to the blow-up of $S^2$ at a finite number of distinct points $R_1, \ldots, R_m$. If $X$ is algebraically diffeomorphic to $S^1 \times S^1$ then $\text{Diff}_{\text{alg}}(X)$ acts $n$-transitively by [1, Theorem 1.3]. Therefore, we may assume that $X$ is algebraically diffeomorphic to the blow-up $B_{R_1, \ldots, R_m}(S^2)$ of $S^2$ at $R_1, \ldots, R_m$. We will show that $\text{Diff}_{\text{alg}}(X)$ acts $n$-transitively on $X$ for all $n$ by induction on $m$.

If $m = 0$, then $\text{Diff}_{\text{alg}}(X)$ is $n$-transitive, for all $n$, by Theorem 2.3. Let $m > 0$, and let $X$ be $B_{R_1, \ldots, R_m}(S^2)$. Let $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ be two $n$-tuples of points of $X$ where $P_j \neq P_k$ and $Q_j \neq Q_k$ whenever $j \neq k$. We want to show that there is an algebraic diffeomorphism $\varphi$ of $X$ such that $\varphi(P_j) = Q_j$ for all $j$. 
We identify $\mathbb{P}^2(\mathbb{R})$ with $B_{R_m}(S^2)$ via an algebraic diffeomorphism. We may consider $R_1, \ldots, R_{m-1}$ as points of $\mathbb{P}^2(\mathbb{R})$ and the surface $X$ is the surface $B_{R_1, \ldots, R_{m-1}}(\mathbb{P}^2(\mathbb{R}))$. Let $\pi: X \to \mathbb{P}^2(\mathbb{R})$ be the blowing-up morphism. Let $L$ be a line in $\mathbb{P}^2(\mathbb{R})$ that does not contain any of the points $R_k, \pi(P_j), \pi(Q_j)$. The inverse image $\tilde{L}$ of $L$ in $X$ is a real algebraic curve in $X$. We show that $\tilde{L}$ is contractible.

Since $\pi$ is an algebraic diffeomorphism from a neighborhood of $\tilde{L}$ in $X$ onto a neighborhood of $L$ in $\mathbb{P}^2(\mathbb{R})$, the inverse image $\tilde{L}$ is a nonsingular rational real algebraic curve contained in $X$. Moreover, since the normal bundle of $L$ in $\mathbb{P}^2(\mathbb{R})$ is nontrivial, the normal bundle of $\tilde{L}$ in $X$ is nontrivial.

A complexification of $\mathbb{P}^2(\mathbb{R})$ is the projective plane $\mathbb{P}^2$. Therefore, a complexification of $X$ is the algebraic variety over $\mathbb{R}$ obtained from $\mathbb{P}^2$ by blowing up the real points $R_1, \ldots, R_{m}$ of $\mathbb{P}^2$. Denote this complexification by $X$. Let $\mathcal{L}$ be the Zariski closure of $L$ in $\mathbb{P}^2$. Of course, $\mathcal{L}$ is a nonsingular algebraic curve over $\mathbb{R}$ whose self-intersection is equal to 1. Denote by $\tilde{\pi}$ the blowing-up morphism from $X$ into $\mathbb{P}^2$, and by $\tilde{\mathcal{L}}$ the inverse image of $\mathcal{L}$ by $\tilde{\pi}$. Since $\tilde{\pi}$ is an isomorphism over a neighborhood of $\mathcal{L}$, the algebraic curve $\tilde{\mathcal{L}}$ over $\mathbb{R}$ is a nonsingular complexification of $\tilde{L}$, and its self-intersection is equal to 1.

It follows from Theorem 3.2 that $\tilde{L}$ is contractible. Let $Y$ be the resulting surface and let $\rho: X \to Y$ be the morphism that contracts $\tilde{L}$ to a point $P$, see Definition 3.1. Let $\sigma: \mathbb{P}^2(\mathbb{R}) \to S^2$ be the morphism that contracts the line $L$ of $\mathbb{P}^2(\mathbb{R})$ to a point. Then $\pi$ induces a morphism $\tau: Y \to S^2$, i.e., one has the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & Y \\
\downarrow{\pi} & & \downarrow{\tau} \\
\mathbb{P}^2(\mathbb{R}) & & S^2 \\
\end{array}
\]

The morphism $\tau$ is the blow-up of $S^2$ at the points $R_1, \ldots, R_{m-1}$. Since the real algebraic curve $\tilde{L}$ does not contain any of the points $P_j$ or $Q_j$ of $X$, the points $\rho(P_1), \ldots, \rho(P_n)$ are distinct, and the same holds for the points $\rho(Q_1), \ldots, \rho(Q_n)$. Moreover, $P \neq \sigma(P_j)$ and $P \neq \sigma(Q_j)$ for all $j$. By the induction hypothesis, the group $\text{Diff}_{\text{alg}}(Y)$ acts $(n+1)$-transitively on $Y$. Therefore, there is an algebraic diffeomorphism $\psi$ of $Y$ such that $\psi(\rho(P_j)) = \rho(Q_j)$ and $\psi(P) = P$. Since $X$ is the blow-up of $Y$ at $P$, the map $\psi$ induces an algebraic diffeomorphism $\varphi$ of $X$ with the required property. □
5. Classification of rational real algebraic surfaces

Proof of Theorem 1.5. Let $X$ and $Y$ be a rational nonsingular compact connected real algebraic surfaces. Of course, if $X$ and $Y$ are algebraically diffeomorphic then $X$ and $Y$ are homeomorphic. In order to prove the converse, suppose that $X$ and $Y$ are homeomorphic. We show that there is an algebraic diffeomorphism from $X$ onto $Y$.

By Theorem 4.3, we may assume that $X$ and $Y$ are not homeomorphic to $S^1 \times S^1$. Then, again by Theorem 4.3, $X$ and $Y$ are both algebraically diffeomorphic to a real algebraic surface obtained from $S^2$ by blowing up a finite number of distinct points. Hence, there are distinct points $P_1, \ldots, P_n$ of $S^2$ and distinct points $Q_1, \ldots, Q_m$ of $S^2$ such that

$$X \cong B_{P_1, \ldots, P_n}(S^2) \quad \text{and} \quad Y \cong B_{Q_1, \ldots, Q_m}(S^2).$$

Since $X$ and $Y$ are homeomorphic, $m = n$. By Theorem 2.3, there is an algebraic diffeomorphism $\varphi$ from $S^2$ into $S^2$ such that $\varphi(P_i) = Q_i$ for all $i$. It follows that $\varphi$ induces an algebraic diffeomorphism from $X$ onto $Y$. $\square$

References


Johannes Huisman, Département de Mathématiques, Laboratoire CNRS UMR 6205, Université de Bretagne Occidentale, 6, avenue Victor Le Gorgeu, CS 93837, 29238 Brest cedex 3, France. Tel. +33 2 98 01 61 98, Fax +33 2 98 01 67 90
E-mail address: johannes.huisman@univ-brest.fr
URL: http://pageperso.univ-brest.fr/~huisman

Frédéric Mangolte, Laboratoire de Mathématiques, Université de Savoie, 73376 Le Bourget du Lac Cedex, France, Phone: +33 (0)4 79 75 86 60, Fax: +33 (0)4 79 75 81 42
E-mail address: mangolte@univ-savoie.fr
URL: http://www.lama.univ-savoie.fr/~mangolte